Discrete Mathematics I — So 2016

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Exercise Sheet 1

Due date: 16:00, Apr 26th, at the end of lecture.
Late submissions will be trampled under the hooves of a Dothraki horde.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Consider the word RHABARBERBARBARA.

(a) How many distinct words can be formed using all 16 letters?

(b) What if we only use 15 out of the 16 letters?

Exercise 2 Recall that for $n \in \mathbb{N}$, $D(n)$ is the set of divisors, and $d(n) = |D(n)|$.

(a) Find an exact expression for $d(n)$ in terms of the prime factorisation of n.

(b) Let $s(n) = \sum_{d \in D(n)} d$ be the *sum* of all of the divisors of n. Find an expression for $s(n)$ in terms of its prime factorisation.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S01.html`.]

Bonus (No points, but lots of respect) Show that there is no odd natural number n for which $s(n) = 2n$.

Exercise 3 Give combinatorial proofs of the following identities.

(a) For all $n \in \mathbb{N}$, $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

(b) For all $r, s \in \mathbb{N}$ and $k \geq 0$, $\sum_{i=0}^{k} \binom{r}{i}\binom{s}{k-i} = \binom{r+s}{k}$.

(c) For all $0 \leq k \leq n$, $\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}$.

Exercise 4 Use mathematical induction to prove the Binomial Theorem, which states that for any real numbers x and y, and any non-negative integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Exercise 5 A multiset of size k is an unordered collection of k elements, with repetition allowed. For instance, $\{1, 2, 3\}$ and $\{1, 1, 2, 3, 3, 3\}$ are different multisets (although they would be the same set, where repetition is not allowed). However, $\{1, 2, 1, 3\}$ and $\{1, 1, 2, 3\}$ are the same multiset.

Show that there are $\binom{n+k-1}{k}$ different multisets of size k using elements from $[n]$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S01.html`.]

Exercise 6 In a knock-out tennis tournament, there are 2^n players and n different rounds. In each round, the surviving players are paired up, with the pairs playing against one another (so if the pair is $\{A, B\}$, the match is A vs B). The winner of each match progresses to the next round, while the loser is kicked out of the tournament. When one player is left, she is crowned as champion.

(a) How many matches will there be before a champion is crowned?

(b) The *draw* is the schedule of the tournament, showing which matches will be played in the first round, and how the winners will progress through the tournament. How many distinct draws are there?

For part (b), note that the draw does *not* include the results of the matches, but just prescribes which winners will play against each other. For example, given the four draws below involving the eight players $\{A, B, C, D, E, F, G, H\}$, the first and the second diagram represent the same draw, but the third one is different because it has different matches in the first round, and the fourth is different because despite having the same first round matches, the winner of A vs B plays a different winner in the second round.

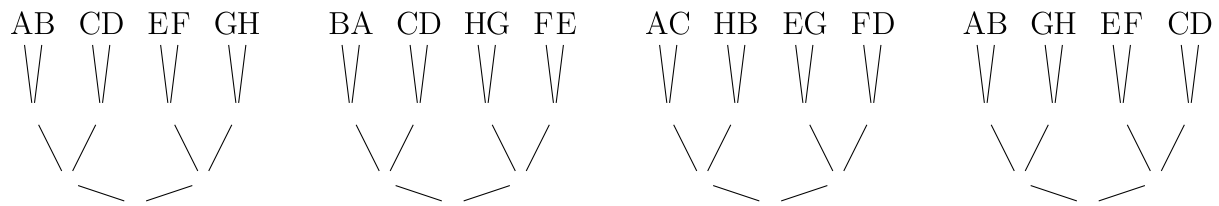

Figure 1: The first two are the same draw, and different from each of the last two.