Discrete Mathematics I So 2016
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Exercise Sheet 10

Due date: 16:00, June 28th, at the end of lecture.
Late submissions will be forced to play as keeper behind a Portuguese defence.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 At the end of the year, the World Tennis Association holds a dinner for the thousands of tennis players who take part in its several tournaments. During these tournaments, the players take part in the doubles' competitions, where they compete with a partner. However, they can change partners for different tournaments, and typically play with several different people over the course of the year. Show that no matter what, there will always be two people at the dinner who have had an equal number of different partners over the course of the year.

Exercise 2 Recall the definition of the Ramsey numbers $R(k, \ell)$ from the previous homework assignment.

(a) Prove that $R(4, 3) \leq 10$.

(b) Show that the bound in (a) can be improved to $R(4, 3) \leq 9$.

(c) Prove that the bound in (b) is tight; that is, $R(4, 3) = 9$.

Exercise 3 Given an n-vertex graph G, the *degree sequence* of a graph is the non-increasing sequence $(d_1, d_2, \ldots, d_n)$, where d_i is the degree of vertex i (ordered by degree). For each of the sequences below, either give a graph with that degree sequence, or prove that no such graph exists.

(a) $(5, 5, 4, 3, 3, 3, 2, 2, 1)$

(b) $(6, 6, 6, 4, 4, 2, 2)$

(c) $(6, 6, 6, 6, 5, 4, 2, 1)$

(d) $(6, 6, 6, 6, 4, 3, 3, 3)$

Exercise 4 Recall the famous Petersen graph from lectures.

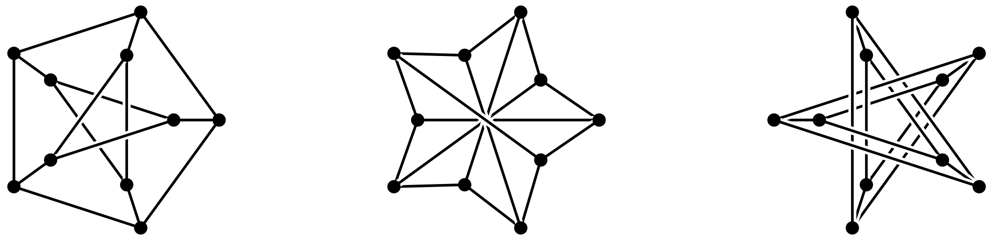
(a) Show that the Petersen graph has at least 120 automorphisms.

(b) Show that the Petersen graph has exactly 120 automorphisms.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S10.html`.]

Bonus (10 pts) Determine the smallest (in terms of the number of vertices) graph that has at least two vertices and whose only automorphism is the identity.

Exercise 5 Determine which pairs of graphs below are isomorphic.



[There's another exercise on the next page.]

Exercise 6 Your goal is to draw a graph G with as few strokes[1] as possible.[2] However, after you draw an edge, you are not allowed to retrace the edge again, so every edge of G must be drawn exactly once.

(a) Prove that if G has $2k$ vertices of odd degree, at least k strokes are necessary.

(b) Show that any connected graph G with exactly $2k$ vertices of odd degree can be drawn with k strokes.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S10.html`.]

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[1]From when you first put your pen on paper to when you next take it off counts as one *stroke*.

[2]"No it's not," you indignantly exclaim! "My goal," you continue, "is to become the greatest combinator that ever lived."

"Surely you do not mean to suggest you could be better than Erdős," we respond.

"Who is this Erdős now," you ask, "and how do you pronounce the 'ő'?"

"You know, from the Erdős–Szekeres theorem," we remind you.[3]

"Oh." You stop and think. "That wasn't such a hard theorem. I bet I could have figured that out on my own."

"Fair enough," we say, "but when you take the Discrete Maths II and III courses, you will hear his name *all the time*. He is one of the founding fathers of combinatorics."

"Fine, whatever," you say, tiring of this conversion, "my goal is to be the best combinator since Erdős. I don't think I'll get there by drawing graphs with as few strokes as possible. And besides," you continue, "even if my goal did involve drawing graphs, I wouldn't try to draw them as efficiently as possible, but rather as carefully and beautifully as possible, because graphs are an important part of my life and they deserve some respect."

You are about to continue when we interrupt.[4] "Very well, it is not *your* goal then. It was, however, the goal of Billy the Kid, the infamous gunslinger from the Old West."

"I find that hard to believe," you scoff. "Why would he have cared about drawing graphs?"

"Why indeed? Well you see," we explain, "one day Billy the Kid rode into town[5] and saw a poster advertising a competition to crown the 'Quickest Draw in the West.' 'Well now,' he thought to himself, 'I do believe that title rightly belongs to me.' So thinking, he made his way to the saloon where the contestants had been instructed to congregate, twirling his pistols along the way to make sure his trigger finger was sharp and ready.

"Imagine his surprise, then," we proceed with our historical tale, "when he discovered that the 'Quickest Draw' in the competition did not refer to how quickly one could remove one's pistol from one's hip and fire at an opponent, but rather to the speed with which one could depict a graph on a piece of paper. A clear-cut case of false advertising if ever there was one, but unfortunately for him the Federal Trade Commission would not be founded until 33 years after his death.

"Still, Billy the Kid was never one to back down from a challenge, and he rightfully figured that rather than drawing each edge one at a time, he could speed up the process by drawing several edges in a single stroke. This led him to wonder how many strokes he would need to draw a graph with $2k$ vertices of odd degree. Eventually, he realised that k strokes would suffice, and one could not do better.[6] After all," we conclude, "they don't call this Billy the Kid's theorem for nothing."[7]

[3]We ignore the second part of your question because it is difficult to resolve pronunciation issues through the medium of footnotes. This is not to suggest that footnotes are not the greatest thing ever, but sadly even footnotes have their limitations.

[4]Not because we don't care about your goals and aspirations (we do), but because we are aware that we are running out of space on this page.

[5]Atop a horse, no doubt, but further equestrian details shall not be forthcoming. For that matter, we shall also not be naming the town.

[6]Unfortunately for him, the competition had ended long before he reached this realisation. Still, he was ready for any future such tournament.

[7]In fact, they don't call this Billy the Kid's theorem at all.