Discrete Mathematics I — So 2016
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Exercise Sheet 11

Due date: 16:00, July 5th, at the end of lecture.
Late submissions will be thrown into a lake by Cristiano Ronaldo.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Prove that there are $2^{\binom{n-1}{2}}$ labelled graphs on n vertices where every vertex has even degree.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S11.html`.]

Exercise 2 Given a graph G on the vertex set $[n]$, the complementary graph $\overline{G}$ has as edges $E(\overline{G}) = \binom{[n]}{2} \setminus E(G)$, so that an edge is present in $\overline{G}$ if and only if it is missing from G.

(a) Show that for any $n \in \mathbb{N}$ and any graph G on n vertices, either G or $\overline{G}$ is connected.

(b) Show that any n-vertex graph with at least $\binom{n-1}{2} + 1$ edges is connected.

Exercise 3 Let G be a connected graph that does not contain P_4 or K_3 as *induced* subgraphs. Prove that G must be a complete bipartite graph.

Exercise 4 Suppose we are given an n-vertex graph G. Set $G_0 = G$. For every $i \geq 0$, if there is a pair of vertices $\{u, v\} \notin E(G_i)$ whose degrees in G_i sum up to n or more, let G_{i+1} be the graph obtained by adding the edge $\{u, v\}$ to G_i. If there is no such pair of vertices, stop this process, and define $\mathrm{cl}(G) = G_i$ to be the *closure* of G.

(a) For $1 \leq a \leq b \leq n-1$ with $a + b = n$, determine $\mathrm{cl}(K_{a,b})$.

(b) Prove that G has a Hamiltonian cycle if and only if $\mathrm{cl}(G)$ does.

Exercise 5 A TV remote accepts two batteries, but will only function if both of the batteries are charged. There are nine batteries, of which only three are charged, and the other six have no charge remaining.[1] However, there is no way to distinguish between the charged and drained batteries other than putting them in the remote and seeing if it works. Determine the minimum number of pairs of batteries that must, in the worst case,[10] be tested in order to get the remote working.[11]

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[1]Your senses having been finally tuned over the course of this semester, you anticipate there is a story behind this scenario. Why in the world would one keep three charged batteries together with six spent ones, instead of disposing[2] of the batteries once they had been used? You are, of course, quite right — this exercise, like so many before it, is drawn from a real-world scenario, which we shall describe below.

After his heartbreaking[3] penalty miss in the Copa America final against Chile, and his subsequent emotional retirement from international football, a terribly unshaven Lionel Messi is to be found on his sofa in front of his TV,[4] a position he has hardly left in days.[5] His family fuss over him, trying to raise his spirits, while Tata Martino tries to get him on the phone, desperate for him to un-retire, yet the only thing Messi seems to want to do is to watch that penalty miss over and over and over again, his morale sinking lower with every replay.

This deleterious effects of this ritual of Messi's are not limited to his sense of self, but are also taxing on the batteries of his remote. With his furious and repeated pressing of the 'play', 'pause' and 'rewind' buttons, the batteries that fuel the remote are quickly depleted. Fortunately for Messi, he has no shortage of batteries at his disposal.

Indeed, he started with a fresh pack of 11 batteries.[6] Once a pair of batteries in his remote stop working, he takes them out and replaces them with some other batteries from the pack. While he knows he should throw the used batteries in the bin, his confidence is so shaken by the penalty miss that he does not trust himself to hit the target. Instead, then, he returns the used batteries to the same pile as the fresh ones.

Once his batteries expire for the fourth time, Messi knows that the two batteries that just died are useless, and there are three fresh batteries and six expired batteries among the other nine batteries.[7] However, he realises that at this point it might take him quite some time to find two working batteries. The prospect of trying every possible pair is almost as repulsive as that of playing for Argentina again, and so he wonders whether there is a quicker way of getting the remote in working order again, leading to the question at hand.

[2]Where by 'disposing', I mean recycling,[8] for we are all responsible environmentally-conscious inhabitants of this planet.

[3]Yet not entirely unexpected, given his record this season.

[4]Before you feel too sorry for him, we should point out that it is a rather luxurious sofa and a very large QHD screen.[9]

[5]The only movements have been the occasional toilet breaks, for the sofa was quite expensive, and he wouldn't want to damage its resale value.

[6]It was one of those "buy ten, get one free" packs designed to make you think you're getting a bargain.

[7]While the penalty miss may lead you to question Messi's mastery of geometry and angles, keeping track of the ridiculous number of goals he has scored has led to some very sharp arithmetic abilities.

[8]Is it just me, or have rechargeable AA batteries gone out of style? I remember they used to be a thing, back when we weighed things with balances and had pigeons carrying our mail.

[9]A cynic might remark that tax evasion has its advantages, but that would be beneath us, and we shall take the high road in these footnotes.

[10]Naturally, since nothing is going Messi's way at the moment.

Exercise 6 The *girth* of a graph G, denoted $g(G)$, is the length of the shortest cycle in G.

(a) Show that if an n-vertex graph is d-regular and has girth $g = 2k + 1$, then

$$n \geq d \sum_{i=0}^{k-1} (d-1)^i + 1.$$

(b) Provide an example to show the bound is tight when $d = 3$ and $g = 5$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S11.html`.]

Bonus (5 pts) Extend the result in part (a) to the case of even girth. That is, give a corresponding lower bound on the number of vertices in a d-regular graph of girth $g = 2k$.

[11] **Epilogue:** Having successfully solved this problem, Messi begins to feel a bit better about life,[12] and realises that he needs help to emerge from this rut he has trapped himself in. He finds his phone somewhere under a pile of batteries, dials a number he knows by heart and, after the connection is established, forgoes the unnecessary introductions. Voice trembling, he manages to say, "I missed."

At the other end of the line, Cristiano Ronaldo, who had been polishing his latest Champions League winner's medal after yet another disappointing tournament with Portugal, consoles his fellow sportsman. "Don't worry, bro," he says, "it happens to the best of us. Just to be clear, by 'best of us' I mean myself, because I am the best. But even I can miss penalties, like in our match against Austria. Who cares about national teams anyway? Get back to training, because El Clasico won't be the same without you. Anyway, I've got to go now, so later, dude," and Ronaldo, eloquence personified, hangs up.

Surprisingly, or perhaps even inexplicably, Ronaldo's words have the desired effect, and Messi is indeed cheered up. After all, Cristiano has a point — if even CR7 can miss penalties, who can begrudge Messi his failure from 11 metres? With this in mind, Messi gets off his sofa and starts to prepare for yet another season of footballing excellence.

[12] The remedial powers of mathematics are well known to anyone who has ever been stuck on a problem before a sudden flash of inspiration illuminates the road to a solution.