Discrete Mathematics I
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So 2016
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Exercise Sheet 12

Due date: 16:00, July 12th, at the end of lecture.
Late submissions will be donated to the Origami restaurant in Phnom Penh.[1]

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 In lecture we had a theorem claiming that for a graph G on n vertices, the following four statements are equivalent:

(1) G is a tree,

(2) G is connected and has $n-1$ edges,

(3) G is acyclic and has $n-1$ edges, and

(4) for every pair of vertices $u, v \in V(G)$, there is a unique $u-v$ path in G.

In class we proved (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). Complete the proof of the theorem by demonstrating the remaining equivalences.

Exercise 2 Given a graph $G = (V, E)$ and two spanning trees $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ of G, show that for every edge $e_1 \in E_1 \setminus E_2$ there is an edge $e_2 \in E_2 \setminus E_1$ such that $T_1' = (V, (E_1 \setminus \{e_1\}) \cup \{e_2\})$ and $T_2' = (V, (E(T_2) \setminus \{e_2\}) \cup \{e_1\})$ are also *both* spanning trees of G.

Exercise 3 Let $G = (V, E)$ be a connected graph on at least two vertices, and $\omega : E \to \mathbb{R}$ a weighting of its edges. For each $v \in V$, let ω_v be the minimum weight of an edge incident to the vertex v. Prove that for every $v \in V$, *every* minimum weight spanning tree of G contains an edge e such that $\omega(e) = \omega_v$.

[1] Which is, disappointingly, not made out of paper.

Exercise 4 Let G_n be the graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ with all edges except $\{1, 2\}$. How many (labelled) spanning trees does G_n have?

Bonus (5 pts) If you could remove one more edge from G_n, which edge should you remove to minimise the number of spanning trees that remain?

Exercise 5 Two friends, Alice and Bob, play a game on a graph G. They alternately choose vertices of the graph, so Alice chooses a vertex v_1, Bob chooses a vertex v_2, Alice chooses a vertex v_3, and so on. The rules stipulate that at every step, the chosen vertices $v_1, v_2, v_3, \ldots, v_k$ must form a path in the graph G. The game ends when no further moves are possible, and the last player to have chosen a vertex wins.

(a) If $G = K_{a,b}$, what should Alice's first move be?

(b) Prove that Bob wins the game if and only if G has a perfect matching.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S12.html`.]

Bonus (0 pts) Write a word problem for this exercise, describing a real-world situation in which such a game might arise.[2]

Exercise 6 Given a graph G, $\delta(G)$ is the minimum degree in G.

(a) Show that any n-vertex graph G with $e(G)$ edges contains a (not necessarily spanning) subgraph H with minimum degree $\delta(H) \geq \frac{e(G)}{n}$.

(b) Let T be a tree with t edges. Show that if G is an n-vertex graph with $e(G) > (t-1)n$, then G contains T as a subgraph.

(c) For infinitely many values of n, construct an n-vertex graph with $\frac{1}{2}(t-1)n$ edges that does not contain *any* tree with t edges.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S12.html`.]

[2]Don't forget to tell us who Alice and Bob really are, and why they are playing the game.