Discrete Mathematics I | So 2016
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Bonus Sheet[1]

Due date: 16:00, July 19th, at the end of lecture.
Late submissions will earn negative marks.

The exercises below are intended to help you test your mastery of the topics covered during the final weeks of lecture. If you have not yet earned enough homework points for the "aktive Teilnahme" credit, you may submit this sheet to be graded, and the points you earn will be added to your total as bonus points. As usual, you may submit in pairs, you should indicate which three solutions you would like graded, and who authored each solution. Each exercise below is worth the standard 10 points.

Exercise 1 Given a graph G, we denote its maximum degree by $\Delta(G)$ and the size of a maximum matching by $\alpha'(G)$.

(a) Show that if G is bipartite, $\alpha'(G) \geq \frac{e(G)}{\Delta(G)}$.

(b) Deduce that any subgraph of $K_{n,n}$ with at least $n^2 - n + 1$ edges has a perfect matching.

(c) Show that the bound in (b) is best possible.

Exercise 2 Let G be a bipartite graph with parts A and B, and that for every subset $S \subseteq A$, $|N(S)| \geq 2\,|S|$. Prove that for every $a \in A$, we can find some set N_a of 2 neighbours of a,[2] such that whenever $a \neq a'$, we have $N_a \cap N_{a'} = \emptyset$.[3]

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S13.html`.]

Exercise 3 Prove that a simple graph G is 2-connected if and only if for every triple (x, y, z) of distinct vertices, G has an x, z-path that passes through y.

[1] Or, "The Last Chance Saloon."
[2] That is, $|N_a| = 2$ and $N_a \subseteq N(\{a\})$.
[3] If we replace every instance of 2 with a k,[4] this is the Polygamy Hall Theorem.
[4] Except the '2' in "Exercise 2," and the '2' in the footnote marker.

Exercise 4

(a) Show that any graph with $\chi(G) = k$ must have at least $\binom{k}{2}$ edges.

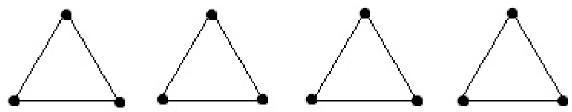
[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S13.html`.]

(b) Deduce that if a graph G is the union[5] of m copies of K_m, then $\chi(G) \leq m^{\frac{3}{2}}$.

Bonus ($500) Show that if the m copies of K_m are *edge*-disjoint, then $\chi(G) = m$.

Exercise 5 Consider the complete graph K_n on the vertices $[n]$. Suppose each vertex $i \in [n]$ is assigned a set of colours S_i. Show that if these sets have the property that for every subset of vertices $I \subseteq [n]$, $|\cup_{i \in I} S_i| \geq |I|$, then there is a proper colouring of K_n where every vertex i receives a colour from its set S_i.

Exercise 6 Let G be the union of four vertex-disjoint triangles, as pictured[6] below.



(a) How many ways are there to properly colour G with the colours red, blue, green and yellow, such that every colour is used at least once?

(b) A set of vertices is called *independent* if it does not contain any edge, and *rainbow* if it has at least one vertex of all of the four colours. Prove that every colouring from (a) contains a rainbow independent set.

[5] Not necessarily disjoint.

[6] Very professionally — thank you, Paint!