Discrete Mathematics I — So 2016
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Practice Sheet

The exercises below offer you the opportunity to practice the material from the last week of lecture,[1] which will be examinable on the second exam. You should not submit anything for these exercises; solutions to the exercises will be posted to the course website at some later date.

Exercise 1 Show that for any graph G and its complement $\overline{G}$, we have $\chi(G) + \chi(\overline{G}) \leq v(G) + 1$.

Exercise 2 A graph is *k-colour-critical* if $\chi(G) = k$, but all proper subgraphs[2] of G have smaller chromatic number. Prove that if G is k-colour-critical, then the Mycielski graph $M(G)$ of G is $(k+1)$-colour-critical.

Exercise 3 A planar graph G is *outerplanar* if there is an embedding of it in the plane such that all vertices are on the boundary of the outer face. Use Kuratowski's Theorem to show that a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

Exercise 4 Prove, without using the Four Colour Theorem, that every outerplanar[3] graph is 3-colourable.

Exercise 5

(a) Prove that every simple planar graph with at least four vertices has at least four vertices of degree less than 6.

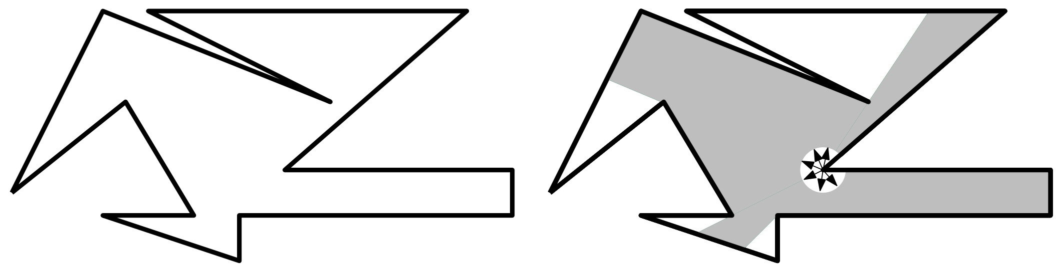
(b) For each even value of n with $n \geq 8$, construct an n-vertex simple planar graph G that has exactly four vertices of degree less than 6.

[1] They will also keep you from getting bored over the summer break.
[2] That is, subgraphs of G where at least one edge or vertex have been removed.
[3] See Exercise 3 for the definition of an outerplanar graph.

Exercise 6

(a) Apply Exercise 4 to prove the Art Gallery Theorem: If an art gallery is laid out as a simple[4] polygon with n sides, then it is possible to place $\lfloor n/3 \rfloor$ guards such that every point of the interior can be watched by some guard.

(b) Construct a polygon that does require $\lfloor n/3 \rfloor$ guards.



An art gallery and what a guard sees from a corner

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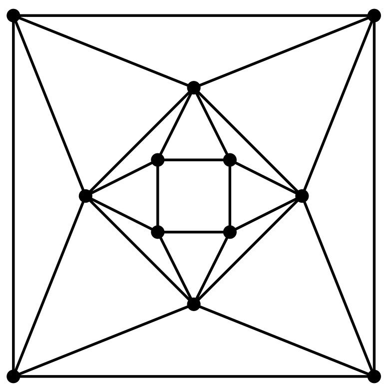
Exercise 7 Define a sequence of plane graphs as follows. Let $G_1 = C_4$. For $n > 1$ obtain G_n from G_{n-1} by adding a new 4-cycle surrounding G_{n-1}, making each vertex of the new cycle also adjacent to the two corresponding consecutive vertices of the previous outside face. The graph G_3 is shown below.



Prove that if n is even, then every proper 4-colouring of G_n uses each colour on exactly n vertices.

[4]But not necessarily convex.

Exercise 8

(a) Give a drawing of K_6 in the real projective plane without any crossings. (Think of the real projective plane as a closed disc where opposite points of the boundary circle are identified.)

(b) Give a drawing of K_7 on the torus without any crossings. (Think of the torus as the unit square $[0,1]^2$, where each boundary point $(0,y)$ is identified with $(1,y)$ and point $(x,0)$ is identified with $(x,1)$.)

Exercise 9

(a) Show that for any graph G, $\chi(G) \leq \Delta(G) + 1$.

(b) The degeneracy $\mathrm{degen}(G)$ of a graph is defined as $\mathrm{degen}(G) = \max_{H \subseteq G} \delta(H)$. Strengthen the bound from (a) by showing that for any graph G, $\chi(G) \leq \mathrm{degen}(G) + 1$.

Bonus From topology, we know that every surface S has its own *Euler characteristic* κ_S.[5] Euler's theorem can be generalised to show that for a map drawn in any surface S, we have

$$V - E + F = \kappa_S,$$

where V is the number of vertices, E the number of edges, and F the number of faces. For example, both the plane and the sphere have characteristic 2, the projective plane[6] has characteristic 1, and the torus[7] has characteristic 0. Generally, κ_S is an integer (possibly negative) less than or equal to 2, and, very loosely, measures how many holes there are in a surface (as well as how orientable it is). Prove that if a simple graph G can be embedded in a surface S with $\kappa_S < 2$, then its chromatic number satisfies

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{49 - 24 \cdot \kappa_S}}{2} \right\rfloor.$$

Observe that Exercise 8 shows this bound is best possible for the projective plane and for the torus.[8] Tantalisingly, the formula gives $\chi(G) \leq 4$ for planar graphs, giving a very short "proof" of the Four Colour Theorem.

[5] Standard notation for the Euler characteristic is χ_S, but we wished to avoid confusion with the chromatic number.

[6] As defined in Exercise 8.

[7] Also defined in Exercise 8.

[8] In fact, this bound is tight on every surface except the Klein bottle.