Discrete Mathematics I | So 2016
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Exercise Sheet 2

Due date: 16:00, May 3rd, at the end of lecture.
Late submissions will be banished to the dark side of the moon.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Prove the following exact formulae for Stirling numbers of the second kind for all integers $n \geq 1$:

(a) $S(n, 2) = 2^{n-1} - 1$.

(b) $S(n, 3) = \frac{1}{6}(3^n - 3 \cdot 2^n + 3)$.

Exercise 2 A professor is getting a room ready for an exam. The room has n desks in one long row, and there will be k students taking the exam. Before the students enter the exam room, the professor wants to place the examination papers on the desks in advance. How many ways can this be done if:

(a) the examination papers are identical?

(b) the examination papers already have the names of the students printed on them?

(c) the examination papers are identical, but the professor wants to have at least two empty desks between each student?

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S02.html`.]

Exercise 3 Show that the average number of cycles in a permutation of length n is exactly H_n, where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S02.html`.]

Exercise 4 Prove the following facts about the Stirling numbers of the first kind.

(a) For every $n \geq 0$ and $x \in \mathbb{C}$,

$$(x+n-1)^{\underline{n}} = \sum_{k=0}^{n} s_{n,k}\, x^k.$$

(b) For every $n \geq 1$, there is some $m(n)$ such that

$$s_{n,0} < s_{n,1} < \ldots < s_{n,m(n)-1} \leq s_{n,m(n)} > s_{n,m(n)+1} > \ldots > s_{n,n}.$$

Moreover, either $m(n) = m(n-1)$ or $m(n) = m(n-1)+1$.

Exercise 5 (FPPOTW[1])

A group[3] of five pirates, called Alice, Bob, Charles, Diana and Erik, have a treasure of 100 identical gold coins that they need to divide between themselves.

(a) How many ways can they divide the coins?

Unfortunately, Alice, Bob, Charles, Diana and Erik do not care about your answer to (a).[4] They will divide the coins according to the traditional rules of piracy. Alice will first suggest a division of the coins - for instance, she might suggest that they each get twenty coins.

Once she has made a suggestion, the pirates (including her) will vote — either they accept the division, or they do not. If a strict majority (strictly more than half) accept the division, then that is how they divide the coins, and the matter is settled.

However, if a majority reject the proposal, or the vote is split evenly, then they kill Alice, and the next pirate (Bob — they proceed alphabetically) makes a proposal instead. They repeat the same process until a division is agreed upon by the remaining pirates.

The pirates all have the following priorities, which they use when deciding how to vote:

1. **Staying alive:** above all, the pirates want to survive — they prefer an outcome where they are alive with 0 coins to one where they die.

2. **Greed:** provided the pirates can stay alive, they want to get as many coins as possible — they prefer an outcome where they are alive with $n+1$ coins to one where they are alive with n coins.

3. **Violence:** all other things being equal, the pirates would like to kill as many other pirates as possible – they prefer an outcome where they are alive with n coins and $k+1$ pirates die to one where they are alive with n coins and k pirates die.

(b) Given these rules, what division should Alice propose?

[1]Fun Pirate Problem of the Week[2]

[2]This probably won't be a weekly thing, because I only know this one fun pirate problem.

[3]In the non-mathematical sense of the word.

[4]After all, they are pirates, not combinators.