Discrete Mathematics I — So 2016

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Exercise Sheet 3

Due date: 16:00, May 10th, at the end of lecture.
Late submissions will be taken to my kennel and fed to my dogs.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Let $q(n)$ denote the number of partitions of n where the two largest parts have the same size.

(a) Prove that $p(n) - p(n-1) = q(n)$.

(b) Prove that $q(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} p(n-k, k)$.

Exercise 2 A partition $\vec{\lambda}$ of n is *self-conjugate* if $\vec{\lambda}^* = \vec{\lambda}$. Let $k(n)$ represent the number of self-conjugate partitions of n. Show that $k(n)$ is equal to the number of partitions of n into distinct odd parts — that is, where each λ_i is odd and no two λ_i are equal.

Exercise 3 For $n \geq 1$, let t_n denote the number of ways of covering[1] a $2 \times n$ rectangle with n tiles of size 2×1.

(a) Find a recurrence relation that the sequence (t_n) satisfies.

(b) Give an explicit formula for t_n.

Exercise 4 Given complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, consider the $n \times n$ matrix

$$M = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{n-1} \end{pmatrix},$$

where $M_{i,j} = \lambda_i^{j-1}$. Prove the Vandermonde formula:

$$\det(M) = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i).$$

[1] You are allowed to rotate the tiles, but you may not perform any surgery. In other words, you may not cut the tiles into smaller pieces and rearrange them.

Exercise 5 In this exercise, you will give a linear algebraic proof of the solution to constant-coefficient linear homogeneous recurrence relations.

Suppose the sequence $(a_n)_{n\geq 0}$ satisfies the recurrence relation

$$a_n = c_{d-1}a_{n-1} + c_{d-2}a_{n-2} + \ldots + c_1 a_{n-d+1} + c_0 a_{n-d},$$

with $c_0 \neq 0$. Suppose further that the characteristic polynomial $P(x) = x^d - c_{d-1}x^{d-1} - \ldots - c_1 x - c_0$ has distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_d$.

(a) Show that the set of sequences satisfying the recurrence relation forms a vector space.[2]

(b) Show that the sequences $(\lambda_i^n)_{n\geq 0}$ are solutions, for every $1 \leq i \leq d$.

(c) Prove that the sequences in (b) form a basis for the space of solutions.

(d) Prove that, for any d initial values $a_n = \alpha_n$ for $0 \leq n \leq d-1$, there is a unique set of coefficients $\beta_1, \beta_2, \ldots, \beta_d$ such that $a_n = \sum_{i=1}^{d} \beta_i \lambda_i^n$ for all $n \geq 0$.

Exercise 6 In this exercise, you will determine the general solution to recurrence relations where repeated roots are allowed.

(a) Suppose the sequence $(b_n)_{n\geq 0}$ has the generating function $B(x) = (1 - \lambda x)^{-m}$. Show that the sequence is given by $b_n = \binom{m+n-1}{n}\lambda^n$.

(b) Let $(a_n)_{n\geq 0}$ be a sequence determined by a recurrence relation with characteristic polynomial $P(x) = \prod_{i=1}^{r}(x - \lambda_i)^{m_i}$; that is, the characteristic polynomial has distinct roots λ_i, each appearing with multiplicity m_i. Show that the solution must take the form[3]

$$a_n = \sum_{i=1}^{r} \left(\sum_{j=1}^{m_i} \beta_{i,j} \binom{n-1+j}{n} \right) \lambda_i^n,$$

where the coefficients $\beta_{i,j}$ can be determined from the initial conditions.

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[2] This would be a subspace of $\mathbb{C}^{\mathbb{N}\cup\{0\}}$, so you only need to show it is closed under linear combinations.

[3] The solution is often presented in the more convenient (but equivalent) form $\sum_{i=1}^{r} \left(\sum_{j=0}^{m_i-1} \tilde{\beta}_{i,j} n^j \right) \lambda_i^n$.