Discrete Mathematics I — So 2016
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Exercise Sheet 4

Due date: 16:00, May 17th, at the end of lecture.
Late submitters will be forced to watch the Twilight Saga.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 In this exercise you will practice building generating functions and decoding their sequences.

(a) Determine a closed form for the generating functions, $A(x)$, of the following sequences.

(i) $a_n = n^3$ for all $n \geq 0$.

(ii) $a_n = \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 2^n + 3^{n/2} & \text{if } n \text{ is divisible by } 4 \\ 2^n - 3^{n/2} & \text{if } n \text{ is even, but not divisible by } 4 \end{cases}.$

(b) Given the following generating functions, find a closed form for the nth term, a_n, of the corresponding sequences.

(I) $A(x) = -\log(1 - 3x^2)$

(II) $A(x) = \cos(x^2)$

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S04.html`.]

Exercise 2 [1] Using the definitions of the derivatives and products of formal power series, show that Leibniz's Rule[3] also holds for formal power series. That is,

$$(F(x) \cdot G(x))' = F'(x) \cdot G(x) + F(x) \cdot G'(x).$$

[1] Of course, this exercise was inspired by the excellent question from lecture[2], and will hopefully convince you that the operations you are used to carrying out on analytic functions carry over to the setting of formal power series.

[2] That being said, I hope you do not feel that asking questions will lead to more homework. You are encouraged to ask any questions you have, and they will only *occasionally* result in homework problems!

[3] Also known as the product rule.

Exercise 3 Prove that the number of partitions of n where every part has odd size is equal to the number of partitions of n without two parts of the same size.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S04.html`.]

Exercise 4 Recall that the Catalan number c_n counts the number of Dyck paths of length $2n$ — that is, diagonal lattice paths from $(0,0)$ to $(2n,0)$ that never go below the x-axis. In this exercise you will give an alternative proof to show $c_n = \frac{1}{n+1}\binom{2n}{n}$.

(a) Determine the total number of diagonal lattice paths from $(0,0)$ to $(2n,k)$ for any integer $k \in \mathbb{Z}$.

(b) Call a diagonal lattice path from $(0,0)$ to $(2n,0)$ *illegal* if it goes below the x-axis. Show there is a bijection between these illegal lattice paths and diagonal lattice paths from $(0,0)$ to $(2n,-2)$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S04.html`.]

(c) Deduce that the number of Dyck paths of length $2n$ is $\frac{1}{n+1}\binom{2n}{n}$.

Exercise 5 One of the reasons that the Catalan numbers are so loved by combinators is that they pop up all over the place.[4] Show that the following sequences are equal to the Catalan sequence $(c_n)_{n\geq 0}$.

(a) $(p_n)_{n\geq 0}$, where p_n is the number of proper nestings of n pairs of parentheses '(' and ')'. A nesting is proper if no parentheses are closed before they are opened, so "()()((()()))" is proper, but "())(()(())()" is not.

(b) $(b_n)_{n\geq 0}$, where b_n is the number of rooted full binary trees with $n+1$ leaves. A rooted full binary tree starts from a root node, and every node either has two descendents (a left child and a right child), or none. If a node has no descendents, it is called a leaf. See Figure 1 for the case $n = 3$.

(c) $(t_n)_{n\geq 0}$, where t_n is the number of triangulations of a convex $(n+2)$-gon; that is, the number of ways a convex polygon with $n+2$ sides can be cut into triangles by connecting its vertices with straight non-crossing lines.[5] See Figure 2 for the case $n = 4$.

[4] Indeed, Richard Stanley's *Enumerative Combinatorics: Volume 2* famously has a set of exercises with no fewer than 66 different Catalan structures!

[5] Note that since there are no convex polygons with 2 sides, we take $t_0 = 1$, since we have nothing to do, and there is one way of doing nothing.

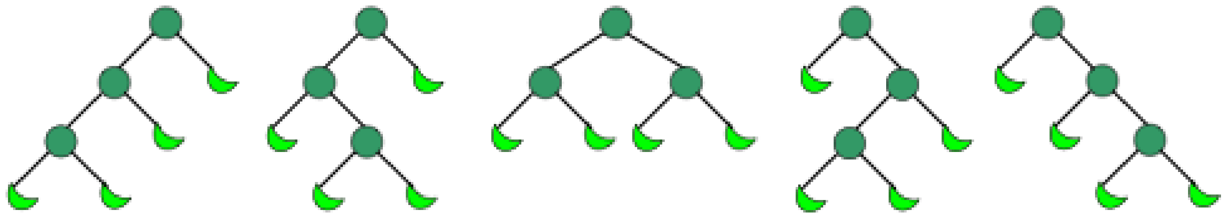

Figure 1: Rooted full binary trees with 4 leaves. (Thanks Wikipedia!)

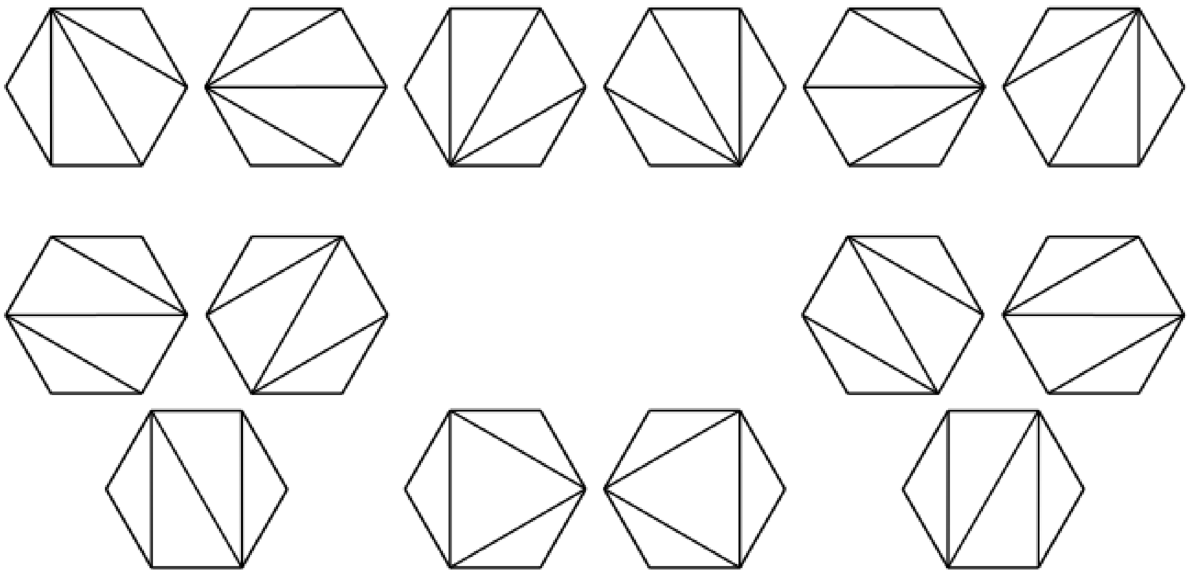

Figure 2: Triangulations of hexagons. (Thanks again, Wikipedia!)