Discrete Mathematics I — So 2016

Tibor Szabó — Codruț Grosu

Shagnik Das — Chris Kusch

Exercise Sheet 5

Due date: 16:00, May 24th, at the end of lecture.
Late submissions will be Germany's next Eurovision contestant.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 In its present incarnation, the United Nations (UN) consists of 193 member states. Within the UN is the United Nations Economic and Social Council (ECOSOC), a non-empty subset of the member states of the UN that, unsurprisingly, deals with economic and social matters.

An unnamed citizen[1], concerned by the sharp rise in the price of McDonalds' ice cream, brings the matter to the attention of the ECOSOC. The ECOSOC agrees that this is a matter of global significance, and decides to form a further subset, the Working Committee to Investigate McDonalds' Pricing of Ice Cream (WCIMPIC) to, well, investigate McDonalds' pricing of ice cream.[2]

(a) Determine the bivariate generating function $A(x,y)$, where the coefficient of $x^n y^k$ counts the number of ways there could be n countries in the ECOSOC, with k of them also in the WCIMPIC.

(b) For historic reasons, the ECOSOC must always have an even number of members. However, in order to ensure they can always have a decisive vote, they decide the the WCIMPIC should consist of an odd number of members. How many ways are there of forming an even ECOSOC containing an odd WCIMPIC?

Exercise 2 Consider the bivariate generating function $B(x,y) = \sum_{n\geq 0} \sum_{k \geq 0} \binom{n}{k} x^n y^k$.

(a) By first evaluating the inner sum, find a closed expression[3] for $B(x,y)$.

(b) By exchanging the order of summation, find the closed form of the coefficient of y^k in $B(x,y)$.

[1]This citizen does, in fact, have a name, but it is withheld from this report to protect her or him from any retaliation by McDonalds.

[2]One must thank the UN for naming their subgroups so well.

[3]That is, a formula not involving sums.

Exercise 3 In this exercise you will supply the combinatorial part of the remarkable Pentagonal Theorem of Euler. Let $\mathcal{E}_n$ denote the set of partitions of n into an even number of distinct parts, and let $\mathcal{O}_n$ denote the set of partitions of n into an odd number of distinct parts.

(a) Given a partition $\vec{\lambda} \in \mathcal{E}_n$, let k denote the number of parts in $\mathcal{E}_n$, so that λ_k is the size of the smallest part in $\vec{\lambda}$. Let $s \geq 1$ be the largest integer such that $\lambda_s = \lambda_1 - s + 1$. If $\lambda_k \leq s$, we can form a new partition by removing the smallest part, and making each of the first λ_k parts bigger by 1. Show that by performing this operation when $\lambda_k \leq s$, and doing something else (you must figure out what!) when $\lambda_k > s$, we get a bijection from $\mathcal{E}_n$ to $\mathcal{O}_n$, unless

   (i) $s = k = \lambda_k$, or

   (ii) $s = k = \lambda_k - 1$.

(b) Deduce that $|\mathcal{E}_n| = |\mathcal{O}_n|$, unless $n = \frac{\ell(3\ell-1)}{2}$ for some (possibly negative) $\ell \in \mathbb{Z}$.

(c) For all $\ell \in \mathbb{Z}$, determine $|\mathcal{E}_n| - |\mathcal{O}_n|$ when $n = \frac{\ell(3\ell-1)}{2}$, and deduce the result of the Pentagonal Theorem.

Exercise 4 Returning to our bookstore example from lecture, suppose you now decide that your bookstores should remain open 24 hours a day, as you anticipate there will be high demand for your books at all hours, day and night. Due to strict labour regulations, you cannot force your employees to work non-stop, and so you must in fact employ the workers in three disjoint shifts, so everyone only works for one continuous stretch of 8 hours. To make things easier administratively, you decide every store should have the shifts from 0000 – 0800, 0800 – 1600, 1600 – 0000, and that everyone will work exactly the same shift every day, so that you only need to makme a schedule for a single day. (However, the labour regulations do not require that you befriend your workers, so from your perspective they are all indistigushable interchangeable drones.)

(a) Given n workers, how many ways are there of distributing them into the three separate shifts? What is the generating function for this sequence?

(b) What generating function counts the number of ways to employ a total of n people in some finite number[4] of bookstores, each operating on a three-shift basis?

(c) If c_n denotes the counting sequence in (b), use partial fractions to deduce that there is some absolute constant γ such that $|c_n - \frac{1}{3} \cdot 2^{n-1}| < \gamma$ for all n.

[4]The number of bookstores is not fixed, but there must always be at least one worker in every store for every shift.

Exercise 5 Given sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, let $\hat{A}(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ and $\hat{B}(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ be their respective exponential generating functions. Let λ, λ_1 and λ_2 be some constants in $\mathbb{C}$. If $\hat{C}(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!}$, determine c_n in terms of $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ in the following cases.

(a) $\hat{C}(x) = \lambda_1 \hat{A}(x) + \lambda_2 \hat{B}(x)$ (b) $\hat{C}(x) = \hat{A}(\lambda x)$ (c) $\hat{C}(x) = x\hat{A}(x)$

(d) $\hat{C}(x) = \hat{A}(x^2)$ (e) $\hat{C}(x) = \frac{d\hat{A}}{dx}$ (f) $\hat{C}(x) = \int_0^x \hat{A}(t)\, dt$