Discrete Mathematics I — So 2016

Tibor Szabó — Codruţ Grosu
Shagnik Das — Chris Kusch

Exercise Sheet 6

Due date: 16:00, May 31st, at the end of lecture.
Late submissions will be used to hold the door.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1

(a) Suppose $\hat{A}(x)$ is the exponential generating function enumerating the number of ways of building a Type I structure on the set $[n]$, and $\hat{B}(x)$ is the exponential generating function counting the number of Type II structures on the set $[n]$, where $b_0 = 0$. Give a combinatorial interpretation for the sequence $(c_n)_{n\geq 0}$ that has exponential generating function $\hat{C}(x) = \hat{A}(\hat{B}(x))$.

(b) The time has come for you to plan your wedding.[1] Of course, the first thing you have to decide is how to sit your n friends at the wedding. You decide that you will split them up into an arbitrary number of groups, with at least 2 people per group. Each group will then sit at a circular table of their own, and you must decide how they will be placed around the table. The only thing that matters is the relative order — who will be sitting to the left and the right of each guest?[5]

Once you have organised each individual table, you then have to decide how to place the table. Each table will be a different distance from the stage where you will be wed, so you have to decide in what order the tables should be placed.

Derive the closed form of exponential generating function $\hat{G}(x)$ that enumerates the number of ways of seating n guests.

[1] You ask, "how did this come to pass?"
One day, while sitting in what seemed like a never-ending Discrete Maths I lecture, you decided to see how much time was left.[2] As your eyes travelled from the board to the clock at the back of the lecture hall, though, they stopped when they came across the most beautiful classmate you had ever seen. As luck would have it, he or she happenned to be looking your way at that very moment, and a connection was made.
You soon started meeting after every lecture, talking about everything there is to discuss: Stirling numbers[3], generating functions[4], number partitions and more. The weeks quickly pass, leading us to where we are today — a few weeks before the wedding.

[2] Of course, while you were hoping there would only be 5 minutes to go, you had closer to 80.

[3] Of both kinds, of course.

[4] You prefer the ordinary kind, but he or she favours the exponential ones.

[5] If you reverse the order around the table (i.e. switch left and right), that counts as a different arrangement.

Exercise 2 Bile Games and Danold Thump decide to invest their considerable savings in two different investment schemes. Let $(g_n)_{n\geq 0}$ and $(t_n)_{n\geq 0}$ respectively denote the amount of money they have after n months. They start out with $g_0 = \$20$ and $g_1 = \$50$, while $t_0 = \$26$ and $t_1 = \$49$.[6]

Their investment schemes promise to provide returns according to the following formulae; for all $n \geq 2$;

$$g_n = 5g_{n-1} - 6g_{n-2} \qquad \text{and} \qquad t_n = t_{n-1} + 2t_{n-2}.$$

(a) In the long run, which investor will be richer?

(b) Suppose we instead had $g_0 = \$25$, and all the other parameters stayed the same. Who would then be richer in the long run?

Exercise 3 Provide the exponential generating functions in closed forms for the following counting sequences.

(a) Set partitions of $[n]$ where every part has size 3 or 5.

(b) Set partitions of $[n]$ where every part has (non-zero) size divisible by 3.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S06.html`.]

Exercise 4 There are n people a pool party,[7] but unfortunately there is only one pool table and only two cues. You want everyone to play exactly one game, so you have to decide which pairs should play against each other, and then prescribe the order in which the $\frac{n}{2}$ games should be played.

(a) If $(p_n)_{n\geq 0}$ counts the number of ways this can be done, find an explicit formula for p_n.[9]

(b) Find the closed form for the exponential generating function $\hat{P}(x)$ corresponding to $(p_n)_{n\geq 0}$.

Exercise 5 In lecture you saw that $n! \geq e\left(\frac{n}{e}\right)^n$.

(a) Show that we also have the bound $n! \leq (n+1)e\left(\frac{n}{e}\right)^n$.

(b) Prove that for all $1 \leq k \leq n$, the binomial coefficient obeys the bounds

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k.$$

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S06.html`.]

---

[6] These numbers have been scaled down by a factor of one billion to ensure the numbers fit on this page.

[7] The kind that involves cue sticks and chalk, not water and swimsuits.[8]

[8] Although this pool party does not have a dress code, so you could wear your swimsuit should you so desire.

[9] Note that if n is odd, you cannot split the people into pairs, so $p_n = 0$.

Exercise 6 Suppose one wishes to do the following: given[10] n children, partition them into two disjoint subsets. The children in the first subset will receive individual mathematics tuition, with every child having the choice of learning either combinatorics or functional analysis (but not both). The second (less fortunate) subset will be food. Of the children in the second subset, two of them will be chosen to have their blood sucked immediately, while the rest will be kept in the freezer for later.[12]

(a) Find the closed form for the exponential generating function $\hat{A}(x)$ that counts the number of ways this can be done.

(b) What is the number of ways of carrying out this procedure when $n = 5$?

---

[10] See Footnote 15 after[11] reading the rest of the problem.

[11] While nothing bad will happen to you if you read it immediately, it will make more sense if you wait.

[12] You might wonder who would wish to do such a thing, for it appears a bit more morbid than the normal hobbies one usually has — football, pottery, or underwater basket weaving. The poor protagonist of this story is Count Calcula. A brilliant combinator from a small town in Hungary,[13] he unfortunately developed some vampiric tendencies after being bitten by a bat[14] that lived in a convex cave. During the day he continues to produce some fantastic mathematics and counts items with aplomb, but when night falls he succumbs to his thirst, and must find some blood to drink.

[13] Which I have been asked not to name, for fear of disrupting their tourism industry. I have also been asked to highlight the fact that they are home to the second-best ice cream store in the entire country.

[14] Of the nocturnal flying variety, not the cricketing kind.

[15] You might also wonder who would give their children to Count Calcula, knowing what fate might befall them. However, the good people of this town are reasonable people, and Count Calcula is famous for his excellent mathematics. Knowing that it is easy to make children,[16] but difficult to produce mathematicians, it is nothing less than a civic duty to offer one's offspring to the Count, in hope that they might get tutored.

[16] The process is not so different from how rabbits are made, as earlier discussed in lecture.