Discrete Mathematics I — So 2016

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Exercise Sheet 7

Due date: 16:00, June 7th, at the end of lecture.
Late submissions will mysteriously vanish, never to be heard of again.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 Someone is planning a round-the-world trip that involves visiting $2n$ cities, with two cities from each of n different countries. He can choose a city to start and end the journey in, with the other $2n - 1$ cities being visited exactly once. However, he has the restriction that the two cities from each country should not be visited consecutively.[1] How many different trips are possible?

(For example, suppose $n = 3$ and the $2n$ cities are {Berlin, Frankfurt, Melbourne, Sydney, New York, Los Angeles}. Berlin → Melbourne → Los Angeles → Sydney → Frankfurt → New York → Berlin is acceptable, but Berlin → Melbourne → Los Angeles → Sydney → New York → Frankfurt → Berlin is not, as the final flight is a domestic one.)

[1]Perhaps you were expecting some Travelling Salesman-type problem, or a Shortest Distance-motivated assignment, for which it would make much more sense to visit neighbouring cities in succession. To understand the restriction from the problem, we must get to know the man behind the tour.

Our world traveller is a boy named Shagnik.[2] Having grown up in Hong Kong, he now lives and works in Berlin, and does all the things you would expect a person in his position to do: he eats Currywurst, he rides the U-Bahn, and he follows Game of Thrones. Despite this vast assortment of varied activities, he only truly feels alive when doing mathematics, playing cricket, or boarding an airplane.

This latter pursuit led to one of his most cherished possessions, his frequent flyer card. The round-the-world trip is not a touristic endeavour, but what is known in the industry as a "mileage run": an attempt to earn as many miles as possible. As such, it is in his interest to take long international flights, rather than short domestic segments.

Now you understand why he must cross a border with every leg of the journey, and may proceed to happily solve the problem.

[2]This character is purely fictional, and any resemblance to actual persons, living or dead, is purely coincidental.[3]

[3]I'm not sure I believe this is actually true, but I was asked[4] to place this character in this assignment, and I shall assume it is.

[4]During last term's Extremal Combinatorics course, I offered any students who received a final grade of 1,0 the chance to create a character (with his or her own backstory) to be used in this term's homework assignments. This is where this Shagnik comes from, and last week's Count Calcula was another such example. I would also like to happily announce[5] the same offer this year: get a 1,0 and win naming rights to a homework character for Winter Semester 2016/17!

[5]It seems appropriate that such an important announcement should be buried in a Level IV footnote.

Exercise 2 When studying the twelvefold ways of counting, we determined that the number of surjective divisions of n distinct items into r distinct parts is $r!S(n,r)$, where $S(n,r)$ is the Stirling number of the second kind. Use the Inclusion-Exclusion Principle to find an expression for $r!S(n,r)$ not involving the Stirling numbers.

Exercise 3 Let n have prime factorisation $n = \prod_{i=1}^{r} p_i^{a_i}$, where $\{p_1, p_2, \ldots, p_r\}$ is the set of distinct prime factors of n.

(a) Show that the Euler totient function is given by $\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)$.

(b) Prove that $n = \sum_{d|n} \phi(d)$, where the sum is over all natural numbers d dividing n.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S07.html`.]

Exercise 4 Let $D(n)$ denote the number of derangements of n elements — that is, permutations of n elements without a fixed point.

(a) Where is the mistake in the following proof?

> Claim: For all $n \geq 2$, $D(n) = (n-1)!$.
>
> Proof: Induction on n. For the base case $n = 2$, $\pi = 21$ is the unique permutation in S_2 without fixed points, so $D(2) = 1 = 1!$.
>
> For the induction step, let $\pi \in S_n$ be a derangement of n elements, and fix some $i \in [n]$. Build a derangement on $n+1$ elements π' by setting
>
> $$\pi'(j) = \begin{cases} \pi(i) & \text{if } j = n+1, \\ n+1 & \text{if } j = i, \\ \pi(j) & \text{otherwise.} \end{cases}$$
>
> This is a permutation of $n+1$ elements without any fixed points: $\pi'(n+1) = \pi(i) \in [n]$, $\pi'(i) = n+1 \neq i$, and $\pi'(j) = \pi(j) \neq j$ for all other j. As there are $D(n)$ options for π and n choices for i, we have $D(n+1) = nD(n)$. By the induction hypothesis, $D(n+1) = n(n-1)! = n!$, as required. □

(b) Prove the recurrence relation $D(n) = (n-1)\left(D(n-1) + D(n-2)\right)$ holds for all $n \geq 2$.

Exercise 5 Suppose we have finite sets $A_1, A_2, \ldots, A_r$. Prove that when k_0 is even,

$$|\cup_{i=1}^r A_i| \geq \sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \subset \binom{[r]}{k}} |\cap_{i \in I} A_i| ,$$

and when k_0 is odd,

$$|\cup_{i=1}^r A_i| \leq \sum_{k=1}^{k_0} (-1)^{k+1} \sum_{I \subset \binom{[r]}{k}} |\cap_{i \in I} A_i| .$$

That is, the partial sums in the Inclusion-Exclusion Principle alternate between upper and lower bounds on the size of the union.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S07.html`.]

Exercise 6 Let $\pi(n) = |\{p \in [n] : p \text{ is prime}\}|$ be the prime number function, counting the number of primes in $[n]$. In this exercise you will determine the order of magnitude of $\pi(n)$.[6]

(a) Show that for every $m \in \mathbb{N}$ and every prime $p \in [m+1, 2m]$, $p | \binom{2m}{m}$.

(b) Deduce $\pi(n) = O\left(\frac{n}{\ln n}\right)$.

(c) Show that if p^k is a prime power such that $p^k | \binom{2m}{m}$, then $p^k \leq 2m$.

(d) Deduce $\pi(n) = \Omega\left(\frac{n}{\ln n}\right)$.

Bonus (0 points) Define $\text{Li}(x) = \int_2^x \frac{dt}{\ln t}$. Prove that $|\pi(n) - \text{Li}(n)| = O\left(n^{\frac{1}{2}} \ln n\right)$.

[6]You are asked to show $\pi(n) = \Theta\left(\frac{n}{\ln n}\right)$. However, more is known. The distribution of the prime numbers has long been central to number theory. Indeed, it was around 1800 that the legendary Legendre conjectured $\pi(n) \approx \frac{n}{\ln n - 1.08366}$. A similar conjecture was made by Gauss around the same time (when he was no older than 16). A few years later, Dirichlet offered the $\text{Li}(n)$ approximation mentioned in the bonus problem.

In 1850, Chebyshev proved that $\frac{n}{\ln n}$ was the correct order of magnitude, and in 1896, Hadamard and de la Vallée Poussin independently extended the work of Riemann and proved the Prime Number Theory, which gives the asymptotics of $\pi(n)$. As conjectured, $\pi(n) \sim \frac{n}{\ln n}$. These proofs all made use of complex analysis.

Since then, several other proofs have been found. Around 1950, Selberg and Erdős found elementary (i.e. not using analysis) proofs. (There was a rather bitter dispute between the two regarding who should get credit for the result.) The simplest proof currently known is due to Newman, although this also uses some complex analysis.