Discrete Mathematics I — So 2016

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Exercise Sheet 8

Due date: 16:00, June 14th, at the end of lecture.
Late submissions will meet the same fate as the Sand Snakes.[1]

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 In this exercise you will prove the claim we used when proving Dilworth's theorem. Recall that we had a finite poset $(P, \leq)$, within which we had a chain C of size $h(P)$ and an antichain A of size $w(P)$ such that $A \cap C = \emptyset$. We defined $A^+ = \{x \in P : x \geq a \text{ for some } a \in A\}$ and $A^- = \{x \in P : x \leq a \text{ for some } a \in A\}$. Prove the following:

(a) $A^+ \cap A^- = A$,

(b) $A^+ \cup A^- = P$, and

(c) $A^+ \setminus A$ and $A^- \setminus A$ are both non-empty.

Exercise 2 Prove that for any finite poset $(P, \leq)$, we have

$$\max\{|C| : C \subseteq P \text{ is a chain}\} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is an antichain partition of } P\}.$$

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S08.html`.]

Exercise 3 Suppose we have a collection of numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $|a_i| \geq 1$ for every $i \in [n]$, and some $x \in \mathbb{R}$.

(a) Show that there are at most $\binom{n}{\lfloor n/2 \rfloor}$ vectors $\vec{\varepsilon} \in \{-1, 1\}^n$ such that

$$\left| x - \sum_{i=1}^{n} \varepsilon_i a_i \right| < 1.$$

(b) For every $n \in \mathbb{N}$, give an example to show that this is best possible: that is, there is some $x \in \mathbb{R}$ and numbers $a_1, \ldots, a_n \in \mathbb{R}$, $|a_i| \geq 1$ for all $i \in [n]$, with exactly $\binom{n}{\lfloor n/2 \rfloor}$ vectors $\vec{\varepsilon} \in \{-1, 1\}^n$ satisfying the above condition.

Exercise 4 The n-divisibility poset $(P_n, \leq)$ has $P_n = [n]$, and $i \leq j$ if and only if $i | j$. Prove that $h(P_n) = \lfloor \log_2 n \rfloor + 1$ and $w(P_n) = \left\lceil \frac{n}{2} \right\rceil$.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S08.html`.]

[1] Which is to say they will be hated and ignored completely.

Exercise 5 Consider the poset $(P, \leq)$ whose Hasse diagram is given in Figure 1.

(a) For every pair $i, j \in [6]$, determine $\mu_{i,j}$.

(b) Prove that for *any* finite poset and all i, j, $\mu_{i,j}$ is always an integer.

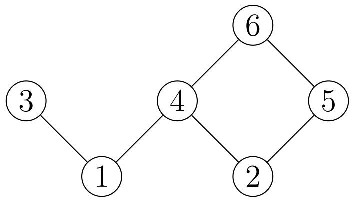

Figure 1: The Hasse diagram of a poset.

Exercise 6 Recall that a chain C in a poset $(P, \leq)$ is a subset $C \subseteq P$ where every pair of elements $x, y \in C$ is comparable. Consider the infinite poset $(2^{\mathbb{N}}, \subseteq)$, where the elements are (possibly infinite) subsets of $\mathbb{N}$, ordered by inclusion.

(a) Show that this poset has a countably infinite[2] chain.

(b) Is there an uncountable[3] chain in the poset?

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S08.html`.]

[2] A set S is countably infinite if there is a bijection from S to $\mathbb{N}$. In other words, you can list the elements $S = \{s_1, s_2, s_3, s_4, \ldots\}$, with every element of S appearing within some finite number of terms.

[3] A set is uncountable if it is infinite, but not countably so. That is, there is no injection from S into $\mathbb{N}$.