Discrete Mathematics I — So 2016
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Exercise Sheet 9

Due date: 16:00, June 21st, at the end of lecture.
Late submissions will be cut up into little pieces and fed to my pigeons.

You should try to solve all of the exercises below, and submit three solutions to be graded — each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each individual solution.

Exercise 1 A family $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of subsets of $[n]$ is said to be *separating* if for any two elements $1 \leq i < j \leq n$, there is some set $F \in \mathcal{F}$ such that $|F \cap \{i, j\}| = 1$; that is, the set F differentiates between i and j.[1]

(a) Prove that the smallest separating family has size $\lceil \log_2 n \rceil$.

The family is said to be *strongly separating* if even more is true: for every $1 \leq i < j \leq n$, there are sets $F, G \in \mathcal{F}$ such that $F \cap \{i, j\} = \{i\}$ and $G \cap \{i, j\} = \{j\}$.[5]

(b) Prove that the smallest strongly separating family has size m, where m is the smallest natural number satisfying $\binom{m}{\lceil m/2 \rceil} \geq n$.[8]

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[1]If one were the type to write long word problems instead of short, concise mathematical exercises, one might picture a fictitious hospital setting. In this make-believe hospital, renowned diagnostician Dr. Grigor Home[2] would be tasked with saving the life of a patient with some mysterious illness. Using his brilliant medical skills, he would have narrowed it down to one of n possible diseases,[3] labelled $1, 2, \ldots, n$. In order to determine which disease the patient had, he would then run a series of tests, where the ith test would return a positive result for the set $F_i \subseteq [n]$ of possible diseases, and a negative result otherwise. The hospital administrator, Dr. Liesel Cuddle,[4] ever keen to cut costs, orders Dr. Home to have as few different tests as possible. The question would then be to determine the minimum number of tests Dr. Home must run to identify the disease. Fortunately for you, however, I am not the type to set such long homework assignments.

[2]Despite his abrasive demeanour, questionable social skills, and addiction to painkillers, Dr. Home is both an extraordinary doctor and (very deep down) good at heart, so he has a team of willing interns and one true friend.

[3]Of course, lupus has already been ruled out.

[4]With whom Dr. Home does not get along with very well, on account of a failed personal relationship.

[5]If one were to continue our hypothetical hypothetical[6] situation, one might imagine that the hospital's lawyer, Mrs. Stay See Warne,[7] steps in. To protect the hospital from litigation, she demands that treatment for disease i can only be started when for every other disease j, there is some test that is positive for i and negative for j. Under these restrictions, how many tests must our stubbled hero Dr. Home now have? However, we will not continue along such imaginative lines.

[6]This is not a typo.

[7]Towards whom Dr. Home is antagonistic, on account of another failed relationship — he gets around.

[8]When n is large, we have $m = \log_2 n + \frac{1}{2} \log_2 \log_2 n + O(1)$.

Exercise 2

(a) Show that any poset with at least $st+1$ elements contains either a chain of size $s+1$ or an antichain of size $t+1$.

(b) Deduce from (a) that any sequence of $st+1$ numbers contains either an increasing subsequence of length $s+1$ or a decreasing subsequence of length $t+1$.

Exercise 3 Let $r_n = F_n \pmod{10}$, where F_n is the nth Fibonacci number. Show that the sequence $(r_n)_{n\in\mathbb{N}}$ is *periodic*; that is, there is some fixed $k \geq 1$ such that $r_n = r_{n+k}$ for all n. What upper bound can you give on the *period* k?

Exercise 4 We can improve our bound on the Ramsey number by considering a two-variable version. Given $k, \ell \in \mathbb{N}$, let $R(k,\ell)$ be the minimum integer N such that any red/blue colouring of the pairs in $\binom{[N]}{2}$ either contains k elements where every pair is coloured red or ℓ elements where every pair is coloured blue.

(a) Show that $R(k,\ell) = R(\ell,k)$ for all $k,\ell \in \mathbb{N}$. Show also that $R(k,1) = 1$ and $R(k,2) = k$.

(b) Prove that for every $k,\ell \geq 2$, $R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1)$.

(c) Deduce that for every $k,\ell \in \mathbb{N}$, $R(k,\ell) \leq \binom{k+\ell-2}{k-1}$.

(d) Conclude that in the symmetric case we have $R(k) = O\left(\frac{4^k}{\sqrt{k}}\right)$, obtaining an improvement by a factor of $\Omega(\sqrt{k})$ on the bound from lecture.

Exercise 5 Define the three-colour Ramsey number $R(k,k,k)$ to be the smallest N such that every red/blue/green colouring of the pairs in $\binom{[N]}{2}$ contains some set of k elements where every pair receives the same colour.

(a) Prove that $R(k,k,k)$ is finite.

(b) Generalising a proof from lecture, prove the lower bound $R(k,k,k) > \frac{k}{3e}\sqrt{3}^k$.

Exercise 6 Consider a 3×7 grid, and suppose each vertex is coloured either black or white. A rectangle[9] inside the grid is said to be *monochromatic* if its four vertices all have the same colour. An example is shown in Figure 1.

(a) Prove that in *every* colouring of the grid, you can always find a monochromatic rectangle.

[9] The rectangle should have distinct vertices and positive area: *degenerate* rectangles with height or width zero are not considered.

(b) Show that this result is best possible, in the sense that there is a black/white colouring of the vertices of a 3×6 grid without any monochromatic rectangles.

(c) How large must n be for every colouring of a $3 \times n$ grid to contain a rectangle whose entire perimeter, not just the four vertices, is monochromatic?

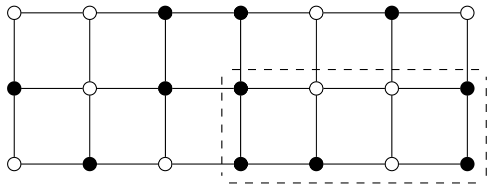

Figure 1: A coloured 3×7 grid with a monochromatic rectangle highlighted.

[Hint at `http://discretemath.imp.fu-berlin.de/DMI-2016/hints/S09.html`.]