Discrete Mathematics I
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So 2016
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Practice Sheet — Solutions[1]

Provided below are possible solutions to the questions from the practice sheet issued towards the end of the course.

Exercise 1 Show that for any graph G and its complement $\overline{G}$, we have $\chi(G) + \chi(\overline{G}) \leq v(G) + 1$.

Solution We prove the statement by induction on the number n of vertices in G. If $n = 1$ then $G = K_1$ and $\chi(G) + \chi(\overline{G}) = 1 + 1$, so the base case is fine.

For the induction step, let $n > 1$. Take an arbitrary vertex $v \in V(G)$, delete it from G and apply induction for $G' = G - v$. By definition $\overline{G - v} = \overline{G} - v$, so

$$\chi(G - v) + \chi(\overline{G} - v) \leq n - 1 + 1 = n.$$

Clearly, $\chi(G) \leq \chi(G - v) + 1$ and $\chi(\overline{G}) \leq \chi(\overline{G} - v) + 1$, since one could always create a proper colouring of G (or $\overline{G}$) by taking an optimal colouring of $G - v$ (or $\overline{G} - v$) and assigning a new colour to v. So

$$\chi(G) + \chi(\overline{G}) \leq \chi(G - v) + 1 + \chi(\overline{G} - v) + 1 \leq n + 2.$$

Since all the numbers involved are integers, if *any* of the inequalities above are strict, we will have the required upper bound of $n+1$. Hence we are done unless $\chi(G) = \chi(G-v)+1$ and $\chi(\overline{G}) = \chi(\overline{G} - v) + 1$, as well as $\chi(G - v) + \chi(\overline{G} - v) = n$.

Note that for any graph H and vertex $u \in V(H)$, if $\deg_H(u) < \chi(H - u)$, then $\chi(H) = \chi(H - u)$, since we can take an optimal colouring of $H - u$, and there will be at least one of the $\chi(H - u)$ colours that is not used on a neighbour of u. We can assign this colour to u to get a proper colouring of H. (Since $H - u \subset H$, we must have $\chi(H) \geq \chi(H - u)$.) Hence if $\chi(H) = \chi(H - u) + 1$, we must have $\deg_H(u) \geq \chi(H - u)$.

This observation provides the necessary contradiction with the three earlier equalities. From the first two equations we get $\deg_G(v) \geq \chi(G - v)$ and $\deg_{\overline{G}}(v) \geq \chi(\overline{G} - v)$, implying

$$n - 1 = d_G(v) + d_{\overline{G}}(v) \geq \chi(G - v) + \chi(\overline{G} - v),$$

which contradicts the third equation. □

[1] Many thanks to Andreas Loos for providing several of these solutions.

Exercise 2 A graph is *k-colour-critical* if $\chi(G) = k$, but all proper subgraphs[2] of G have smaller chromatic number. Prove that if G is k-colour-critical, then the Mycielski graph $M(G)$ of G is $(k+1)$-colour-critical.

Solution A graph G is k-colour-critical if $\chi(H) < \chi(G) = k$ for every proper subgraph H of G. Let $M(G)$ be the Mycielski of a k-colour-critical graph G on vertex set $V(G) = \{v_1, \ldots, v_n\}$. That is, the graph with $V(M(G)) = V(G) \cup \{u_1, \ldots, u_n, w\}$ and $E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}\}$. We know from the lecture that $\chi(G) = k$ implies $\chi(M(G)) = k+1$.

Since there are no isolated vertices in $M(G)$, it is enough to check that $M(G) - e$ is k-colourable for every edge $e \in E(M(G))$. There are three cases.

<u>Case 1:</u> $e = v_i v_j$ for some $1 \leq i < j \leq n$. Since G is colour-critical, we can colour $G - e$ properly with $k-1$ colors, say 1 up to $k-1$. We then colour the vertices $u_1, \ldots, u_n$ with k, and colour w with 1. This is a proper k-colouring of $M(G) - e$.

<u>Case 2:</u> $e = v_i u_j$ for some $1 \leq i \neq j \leq n$. By the definition of Mycielski's construction, we have $v_i v_j \in E(G)$. Now consider $H = G - v_i v_j$. Since G is k-colour-critical, H is $(k-1)$-colourable. So, $M(H)$ is k-colourable by the theorem in the lecture. Moreover, we can see that $M(G) - e = M(H) + v_i v_j + v_j u_i$. The idea is to first properly colour $M(H)$ with k colours, and modify this colouring to obtain a proper k-colouring of $M(G) - e$.

Here is an explicit method. First colour V by $k-1$ colors, say 1 up to $k-1$, according to a proper $(k-1)$-colouring of H. Then, for each $\ell \in \{1, \ldots, n\}$, colour $u_\ell \in U$ with the colour used for $v_\ell \in V$. Finally, colour w with k. This is a proper k-colouring of $M(H)$.

Now we add $v_i v_j$ and $v_j u_i$ to $M(H)$ so that the result will be $M(G) - e$. Then change the colour of v_j to k. Since the colour k was not used in $U \cup V$, this colouring is still proper. Thus, we obtain a proper k-colouring of $M(G) - e$.

<u>Case 3:</u> $e = u_i w$ for some $i = 1, \ldots, n$. First consider the graph $G - v_i$. Since G is colour-critical, $G - v_i$ is $(k-1)$-colourable. Now, in $M(G) - e$, we color $V \setminus \{v_i\}$ by $k-1$ colours, say 1 up to $k-1$, according to a proper $k-1$-colouring of $G - v_i$. Next, for each $\ell \in \{1, \ldots, n\} \setminus \{i\}$ we colour u_ℓ with the colour used for v_ℓ. Then, we can colour v_i, u_i, w with k. This gives a proper k-colouring of $M(G) - e$ since $\{v_i, u_i, w\}$ is an independent set in $M(G) - e$, and the colour k is not used on the other vertices.

To summarise, in each of the three cases we can properly k-colour $M(G) - e$, showing that Mycielski's construction preserves colour-criticality. $\square$

[2] That is, subgraphs of G where at least one edge or vertex have been removed.

Exercise 3 A planar graph G is *outerplanar* if there is an embedding of it in the plane such that all vertices are on the boundary of the outer face. Use Kuratowski's Theorem to show that a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

Solution Kuratowski's Theorem says that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Let G be an outerplanar graph and fix an embedding such that all vertices are on the boundary of the outer face. Construct a graph G' by adding a new vertex v in the outer face and connecting it to all vertices of G. This is a plane drawing of G', which is thus planar and hence does not contain a subdivision of K_5 or $K_{3,3}$. If G contained a subdivision of K_4 or $K_{2,3}$, then this subdivision together with v would give a subdivision of K_5 or $K_{3,3}$ in G', contradicting Kuratowski's Theorem.

Conversely, let G be a graph not containing a subdivision of K_4 or $K_{2,3}$. Let G' be the graph formed by adding a vertex v and connecting it to all vertices of G. Now G' does not contain a subdivision of K_5 or $K_{3,3}$, since otherwise removing v from this subdivision would result in either a subdivided K_4 or a subdivided $K_{2,3}$ in G. Hence G' is planar and we can consider an embedding of G' in the plane without intersections of the edges. If v does not lie on the boundary of the outer face, we can invert the drawing in one of the faces neighbouring v (project the drawing onto the sphere, and then project back onto the plane from such a face) to make one of the faces next to v the outer face. If we remove v, then all vertices of G are on the boundary of the outer face (because they were connected to v in an embedding without intersections of edges) and thus G is outerplanar. $\square$

Exercise 4 Prove, without using the Four Colour Theorem, that every outerplanar[3] graph is 3-colourable.

Solution We prove the statement by induction on the number of vertices n. For $n \leq 3$ every n-vertex graph is 3-colourable. For the induction step, let G be an outerplanar graph. If we have more than one connected component, apply the induction hypothesis and 3-colour each component. Otherwise fix an embedding of G such that all vertices are on the boundary of the outer face.

Let v be an arbitrary vertex. Since we have at least 4 vertices the degree of v is at least 2 (it has 2 neighbours on the outer face). If it is precisely 2, we remove v and connect both neighbours. We get an embedding of an outerplanar graph G' with one fewer vertex. By the induction hypothesis we can 3-colour G'. To get a colouring for G we colour v by the colour not used by either of its neighbours, which is therefore a proper 3-colouring of G.

If the degree of v is at least 3, let u be a neighbour of v which is not next to v on the boundary of the outer face. We split G along the edge uv, obtaining two outerplanar graphs that share precisely the vertices v and u, each with smaller number of vertices. By the induction hypothesis, we can 3-colour both by induction hypothesis. Permuting the colours in one of the subgraphs, we can ensure v and u have the same colours in both graphs, and thus join the colourings together to get a proper colouring of G. □

[3]See Exercise 3 for the definition of an outerplanar graph.

Exercise 5

(a) Prove that every simple planar graph with at least four vertices has at least four vertices of degree less than 6.

(b) For each even value of n with $n \geq 8$, construct an n-vertex simple planar graph G that has exactly four vertices of degree less than 6.

Solution

(a) Let G be a simple planar graph with $n \geq 4$ vertices. Add edges to G, maintaining planarity, until for every $e \notin E(G)$, $G + e$ is not planar. If this graph has 4 vertices with degree less than 6, then the original graph does too.

Now every vertex in G has degree at least 3. Indeed, if there would be a vertex of degree 0 or 1, we could immediately add an edge from it to a vertex on the boundary of the face it is in, still leaving the graph planar. If there is a vertex of degree 2, it is contained on the boundary of two adjacent faces. Since we have at least 4 vertices, at most one of them can be a triangle giving us the possibility to add another edge. Hence we may assume every vertex has degree at least 3.

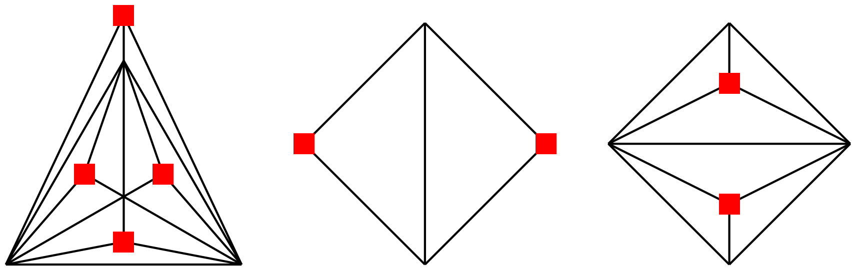
Now suppose for contradiction we only have 3 vertices with degree less than 6. Then, with m being the number of edges in G, we have 3 vertices of degree at least 3 and $n - 3$ vertices of degree at least 6, giving

$$2m = \sum_{v \in G} \deg(v) \geq (n-3) \cdot 6 + 3 \cdot 3 = 6n - 9.$$

However, we know from the corollary of Euler's Theorem that in any planar graph with $n \geq 3$ we have $m \leq 3n - 6$, and so $2m \leq 6n - 12 < 6n - 9$. This gives us a contradiction, and so G must have at least 4 vertices of degree less than 6.

(b) A construction is given below. On the left is the case $n = 8$: an 8-vertex graph with 4 vertices of degree 6 and 4 (red) vertices of degree 3.

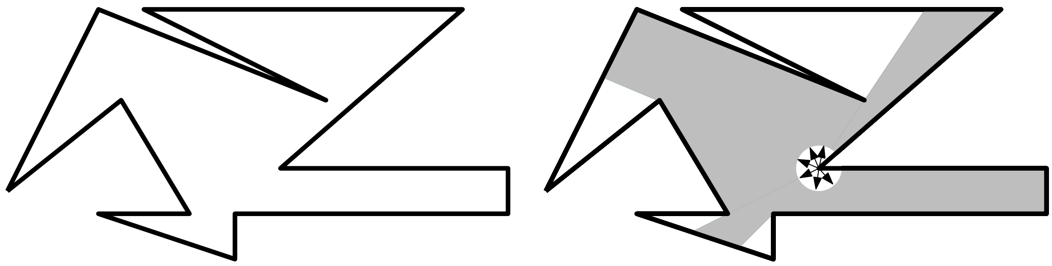
Starting from this construction, we can build one of any larger even size. Note that between any two of the red vertices, we see a C_4 with a chord separating the red vertices, as shown in the middle diagram. We can then replace this C_4 with the graph shown on the right, which increases the degree of the old red vertices to 6, and introduces two new red vertices with degree 3. This have the same C_4 between them, so we can repeat this process ad infinitum. $\square$



Exercise 6

(a) Apply Exercise 4 to prove the Art Gallery Theorem: If an art gallery is laid out as a simple[4] polygon with n sides, then it is possible to place $\lfloor n/3 \rfloor$ guards such that every point of the interior can be watched by some guard.

(b) Construct a polygon that does require $\lfloor n/3 \rfloor$ guards.



An art gallery and what a guard sees from a corner

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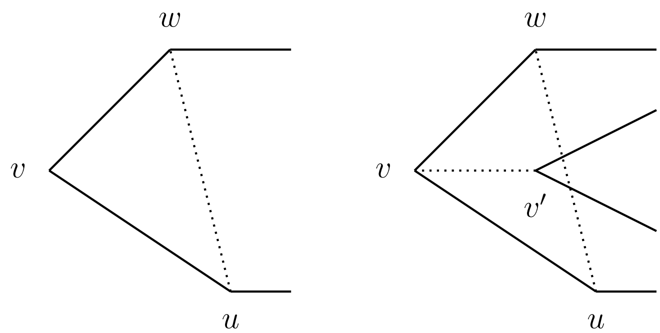
Solution

(a) Take a simple polygon P with $n \geq 3$ sides and vertices. A diagonal is a non-intersecting line segment connecting two vertices of P that is fully contained inside P. A maximal set of non-intersecting diagonals is called a triangulation of P.

First we prove that there always exists a triangulation. For $n = 3$ we only have one triangle. For $n > 3$ we need to find one diagonal. Let v be the leftmost (according to the x-coordinate) vertex of P and u,w be its neighbours. If the line segment from u to w is a diagonal we can add it, and then triangulate the rest by induction.



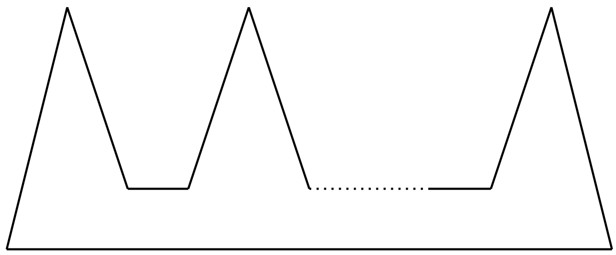
[4]But not necessarily convex.

Otherwise let v' be the leftmost vertex inside the triangle uvw, as pictured above. The line segment from v to v' is then a diagonal, and we are again done by induction.

We proved implicitly that in a triangulation all internal faces are triangles, because otherwise we can apply the induction step to any non triangular face, giving us a new diagonal contradicting the maximality.

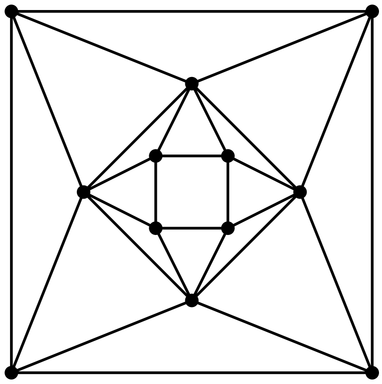
Now we can view the triangulated polygon P as an embedding of a graph G into the plane with all vertices on the boundary of the outer face, i.e. G is outerplanar. By Exercise 4 we know that G is 3-colourable. Every internal face of the embedding is a triangle, so it has a vertex of every colour, and that vertex sees the entire triangular face. Thus every colour class is a valid set of guards, including the smallest class, whose size is at most $\lfloor n/3 \rfloor$.

(b) This is an example of a polygon achieving this bound.



Note that there are $n/3$ triangular "rooms" in this gallery (extending the lines from the top vertices to the bottom line), and the apex of each triangle is only visible from within these rooms. Since these triangles are disjoint, this means that at least $n/3$ guards are needed just to see each top vertex, and hence one cannot guard the polygon with fewer guards. □

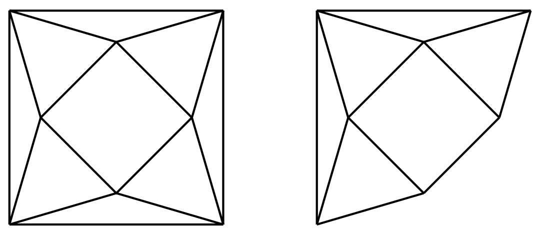
Exercise 7 Define a sequence of plane graphs as follows. Let $G_1 = C_4$. For $n > 1$ obtain G_n from G_{n-1} by adding a new 4-cycle surrounding G_{n-1}, making each vertex of the new cycle also adjacent to the two corresponding consecutive vertices of the previous outside face. The graph G_3 is shown below.



Prove that if n is even, then every proper 4-colouring of G_n uses each colour on exactly n vertices.

Solution Since G_n is planar, we know by the Four Colour Theorem that there exists a proper 4-colouring of G_n. If we look at any two consecutive 4-cycles in G_n they form a subgraph which is isomorphic to G_2, and so it suffices to show that in any proper 4-colouring of G_2, each colour is used exactly twice.

Suppose for contradiction that a colour is used at most once. By removing its colour class, we are left with a 3-colourable subgraph of G_2 with only vertex removed. Since G_2 is vertex-transitive (we can invert G_2, exchanging the inner and outer cycles), we may assume the bottom-right vertex is removed, as shown below.



However, not that the three vertices in the bottom-left triangle, together with the top vertex, form a K_4 with one edge removed. Hence in any proper three-colouring, the endpoints of the missing edge, namely the top vertex and the bottom vertex of the square, must have the same colour. By the same argument, the top vertex and the right vertex of the square also have the same colour. This implies that the bottom and right vertices of the square share a colour, but they are adjacent, which contradicts this subgraph having a proper three-colouring. Hence every colour must appear at least twice in a proper four-colouring of G_2. Since there are only 8 vertices in total, this means each colour appears exactly twice, completing the proof. □

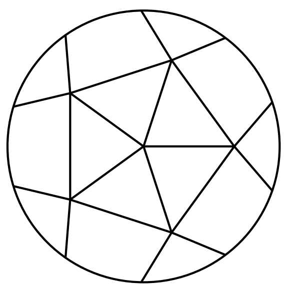
Exercise 8

(a) Give a drawing of K_6 in the real projective plane without any crossings. (Think of the real projective plane as a closed disc where opposite points of the boundary circle are identified.)

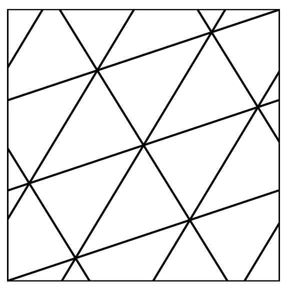
(b) Give a drawing of K_7 on the torus without any crossings. (Think of the torus as the unit square $[0,1]^2$, where each boundary point $(0,y)$ is identified with $(1,y)$ and point $(x,0)$ is identified with $(x,1)$.)

Solution The required drawings are given below.

(a)



(b)



$\square$

Exercise 9

(a) Show that for any graph G, $\chi(G) \leq \Delta(G) + 1$.

(b) The degeneracy $\mathrm{degen}(G)$ of a graph is defined as $\mathrm{degen}(G) = \max_{H \subseteq G} \delta(H)$. Strengthen the bound from (a) by showing that for any graph G, $\chi(G) \leq \mathrm{degen}(G) + 1$.

Solution

(a) We will prove $\chi(G) \leq \Delta(G) + 1$ by induction on the number n of vertices. If $n = 1$, then $G = K_1$, and $\chi(K_1) = 1 = \Delta(K_1) + 1$.

For the induction step, let $v \in V(G)$ be an arbitrary vertex, and let $G' = G - v$. Since G' is a subgraph of G, we have $\Delta(G') \leq \Delta(G)$. By induction, there is a proper colouring of G' using $\Delta(G) + 1$ colours. Since v has at most $\Delta(G)$ neighbours, at least one of the colours is not used on the neighbourhood of v. Assigning that colour to v gives a proper colouring of G, and hence $\chi(G) \leq \Delta(G) + 1$, as required.

(b) With a couple of observations, we can repeat the same inductive proof to show $\chi(G) \leq \mathrm{degen}(G) + 1$.

First note that rather than letting v be an arbitrary vertex, we can find a vertex $v \in V(G)$ of degree at most $\mathrm{degen}(G)$. Indeed, let v be a vertex of minimum degree in G. We then have $\deg(v) = \delta(G) \leq \max_{H \subseteq G} \delta(H)$, since $H = G$ is included in the domain of maximisation.

Secondly, observe that if $G' \subseteq G$, then $\mathrm{degen}(G') \leq \mathrm{degen}(G)$. Indeed, suppose $H' \subseteq G'$ is a subgraph with $\delta(H') = \mathrm{degen}(G')$. We then have $H' \subseteq G' \subseteq G$, and so $\mathrm{degen}(G) = \max_{H \subseteq G} \delta(H) \geq \delta(H') = \mathrm{degen}(G')$. This shows that in the induction step, we can find a proper colouring of G' using $\mathrm{degen}(G) + 1$ colours.

We can now repeat the inductive proof, getting a proper colouring of G with at most $\mathrm{degen}(G) + 1$ colours. In closing, we remark that since $\mathrm{degen}(G) = \max_{H \subseteq G} \delta(H) \leq \max_{H \subseteq G} \Delta(H) \leq \Delta(G)$, this does indeed strengthen the result from (a). $\square$

Bonus The bonus exercise is left open as a challenge for the interested reader.