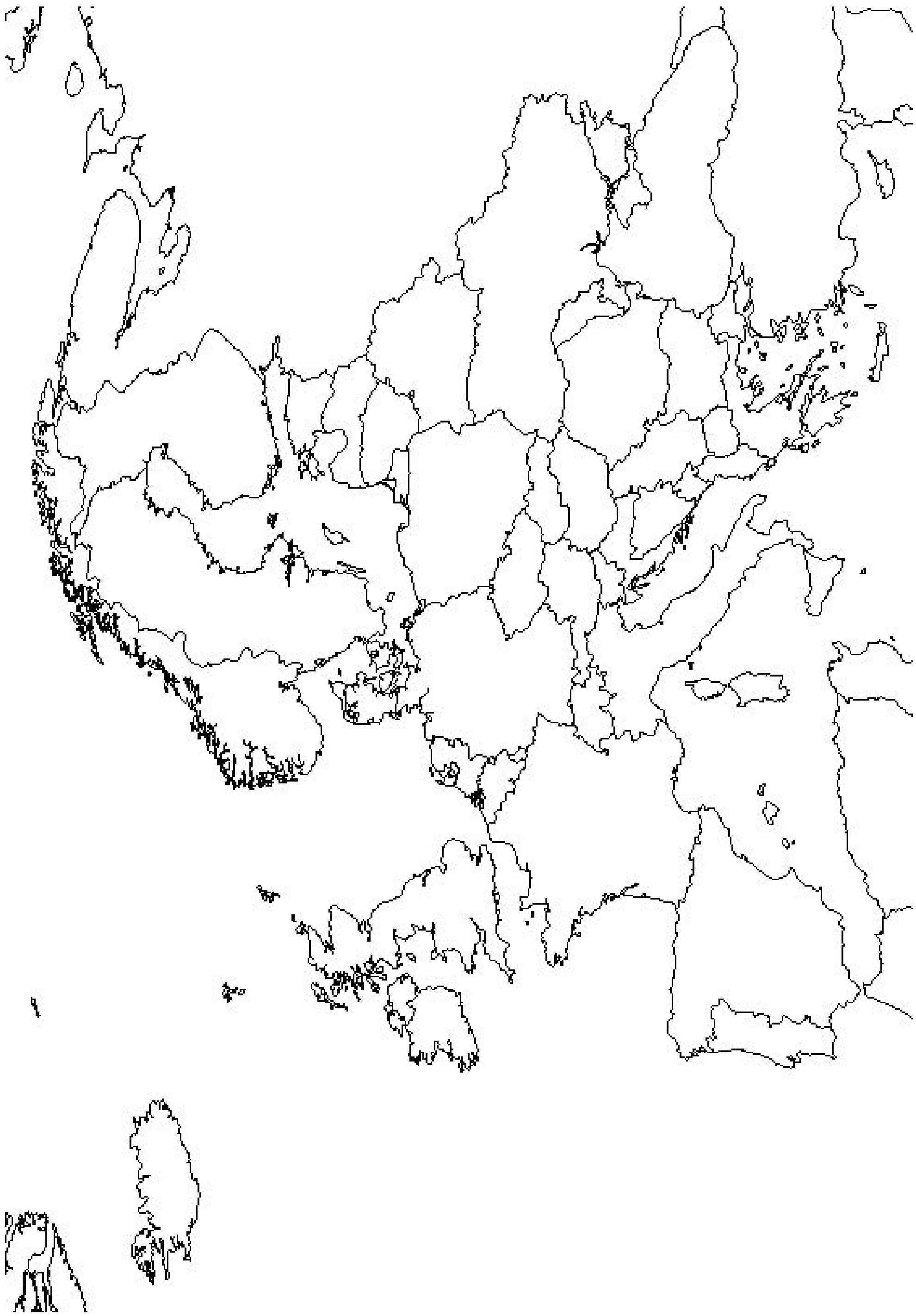
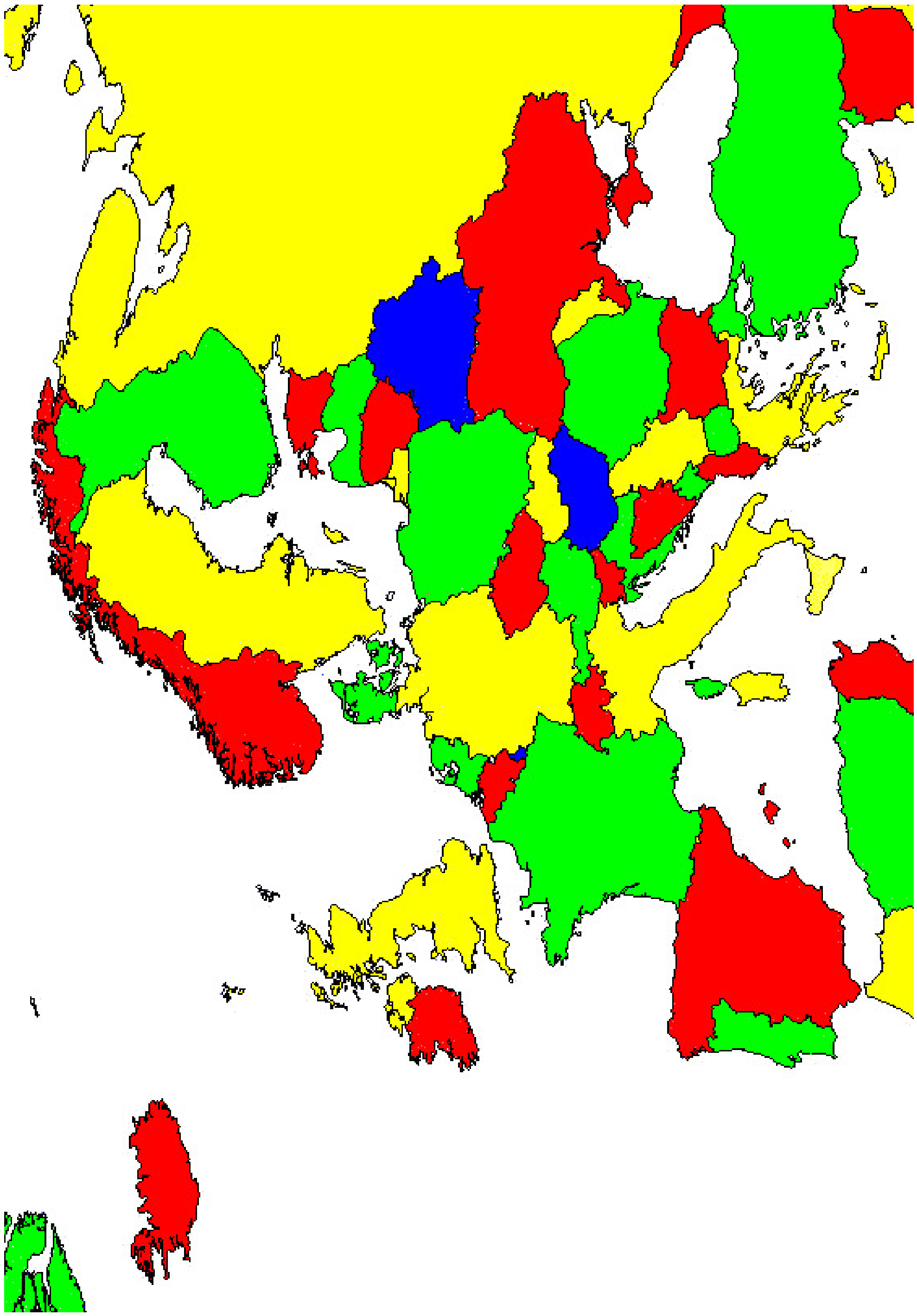
How many colors are needed to color a map?



Is 4 always enough?



Two relevant concepts

How many colors do we need to color a map so neighboring countries get different colors?
Simplifying assumption (not true in reality): Each country is bounded by a simple continuous curve.

Auxiliary graph: $V(G) =$ set of countries, $E(G) =$ pairs of countries that are neighboring (share a 1-dimensional piece of their boundary. (just points are not enough!)

Graph colorings: We then want a coloring of the *vertices* of this auxiliary graph, such that adjacent vertices receive distinct colors.

Planar graphs: The auxiliary graph G of the map has a special property: it can be drawn into the plane such that the edges do not cross. Indeed: draw the vertex representing the country in the “middle” (the “capitol”) and draw a curve to the middle of the boundary curve with each country. This drawing forms an embedding of the graph G in the plane so that the edges do not intersect.

Vertex coloring, chromatic number

A k-coloring of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are called colors; the vertices of one color form a color class.

A k-coloring is proper if adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring.

The chromatic number is

$$\chi(G) := \min\{k : \ G \text{ is } k\text{-colorable}\}.$$

A graph G is k-chromatic if $\chi(G) = k$.

Examples. K_n, $K_{n,m}$, C_5, Petersen

A graph G is k-color-critical (or k-critical) if $\chi(H) < \chi(G) = k$ for every *proper* subgraph H of G.

Characterization of 1-, 2-, 3-critical graphs.

Lower bounds

Simple lower bounds

$$\begin{aligned}\chi(G) &\geq \omega(G) \\ \chi(G) &\geq \frac{v(G)}{\alpha(G)}\end{aligned}$$

Examples for $\chi(G) \neq \omega(G)$:

- odd cycles of length at least 5,

$$\chi(C_{2k+1}) \geq \frac{v(C_{2k+1})}{\alpha(C_{2k+1})} = 2+\frac{1}{k} > 2 = \omega(C_{2k+1})$$

- complements of odd cycles of order at least 5,

$$\chi(\overline{C}_{2k+1}) \geq \frac{v(\overline{C}_{2k+1})}{\alpha(\overline{C}_{2k+1})} = k+\frac{1}{2} > k = \omega(\overline{C}_{2k+1})$$

- random graph $G = G(n, \frac{1}{2})$, almost surely

$$\chi(G) \approx \frac{n}{2\log n} > 2\log n \approx \omega(G)$$

Mycielski's Construction

The bound $\chi(G) \geq \omega(G)$ could be arbitrarily bad.

Construction. Given graph G with vertices $v_1, \ldots, v_n$, we define supergraph $M(G)$.

$V(M(G)) = V(G) \cup \{u_1, \ldots u_n, w\}$.

$E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}\}$.

Theorem.

(i) If G is triangle-free, then so is $M(G)$.

(ii) If $\chi(G) = k$, then $\chi(M(G)) = k + 1$.

Upper bounds $\chi(G) \leq \Delta(G) + 1$.

Proof. Algorithmic. Greedy coloring.

Jordan Curves

For continuous $\gamma : [0,1] \to \mathbb{R}^2$, the subset

$$Im(\gamma) := \{\gamma(x) : x \in [0,1]\} \subseteq \mathbb{R}^2$$

is called a curve. $\gamma(0)$ and $\gamma(1)$ are called the *end-points* of the curve.

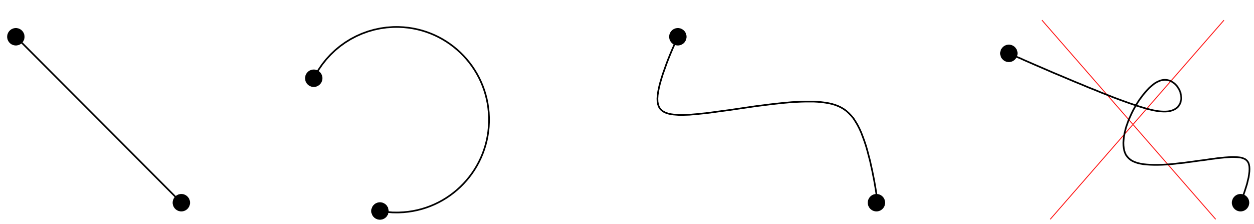
A curve is closed if $\gamma(0) = \gamma(1)$.
A curve is simple if it is injective (except possibly $\gamma(0) = \gamma(1)$).

A closed simple curve is called a Jordan-curve.

Examples: Line segments between $p, q \in \mathbb{R}^2$

$$x \mapsto xp + (1-x)q \ ,$$

circular arcs, Bezier-curves without self-intersection are simple curves*



*The Peano curve is not simple.

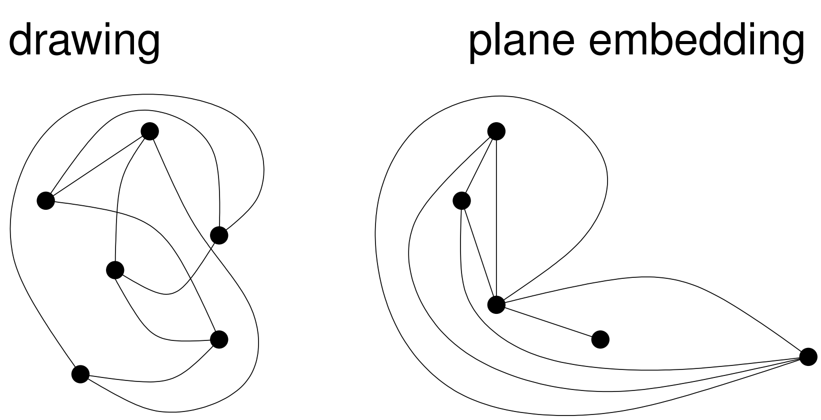
Drawing of graphs

A drawing of a multigraph G is a function f defined on $V(G) \cup E(G)$ such that

- $f|_{V(G)} : V(G) \to \mathbb{R}^2$ is injective and
- $f(uv)$ is an $f(u), f(v)$-curve for every $uv \in E(G)$.

A point in $f(e) \cap f(e')$ that is not a common endpoint of e and e' is called a crossing.

A multigraph is planar if it has a drawing without crossings. Such a drawing is a planar embedding of G. A planar (multi)graph *together* with a particular planar embedding is called a plane (multi)graph.



Are there non-planar graphs?

Proposition. K_5 and $K_{3,3}$ cannot be drawn without crossing.

Proof. Define the *conflict graph* of edges.

The unconscious ingredient.

Jordan Curve Theorem. If C is a Jordan curve, then $\mathbb{R}^2 \setminus C$ is the disjoint union of two nonempty path-connected* open† sets‡. One of these is bounded (the *interior* of C), the other is unbounded (the *exterior* of C) and both have C as their boundary.

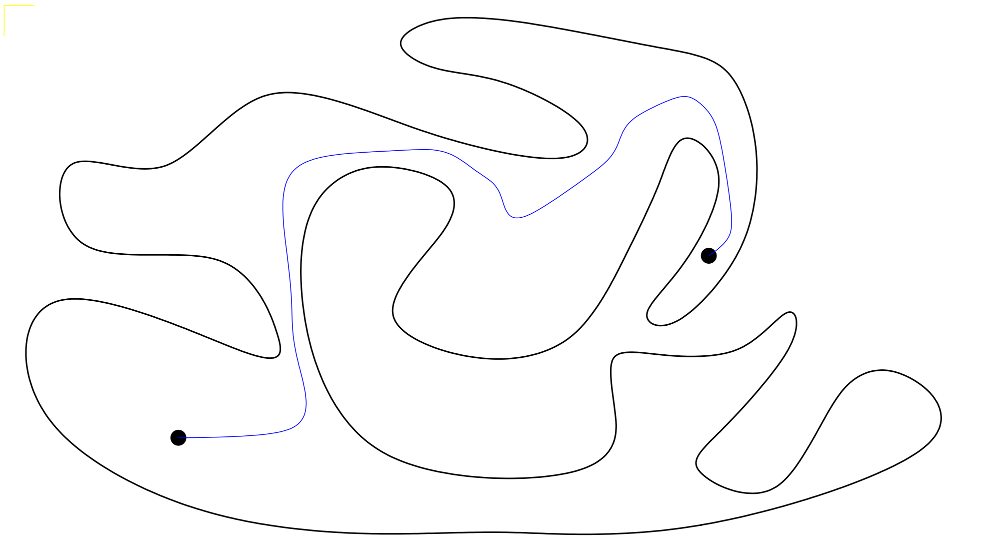
Remark Every curve between a point of the exterior and a point of the interior contains a point of C.

$^*U \subseteq \mathbb{R}^2$ is path-connected if for $\forall u, v \in U\ \exists$ a u, v-curve in U

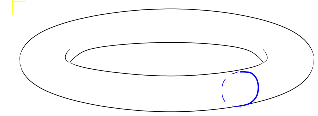
$^\dagger U \subseteq R^2$ is an open if for every $p \in U$ there is an $\epsilon > 0$ such that the disk of radius ϵ with center p is contained in U.

‡A path-connected open set is called a region.

Jordan Curve Theorem



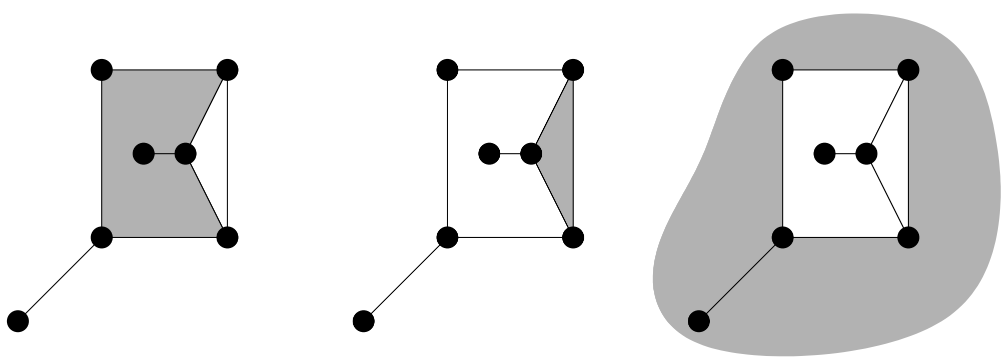
JCT is not true on the torus!



Faces

The faces of a plane multigraph are the maximal regions of the plane that contain no points used in the embedding.

A finite plane multigraph G has one unbounded face (also called outer face).



Proposition If G is a connected plane multigraph with $\delta(G) \geq 2$ then the boundary of each face forms a cycle in G. If $l(F_i)$ denotes the length of face F_i then

$$2e(G) = \sum l(F_i).$$

Euler's Formula

Theorem.(Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Proof. Induction on e.

Base Case. If $e = 0$, then $n = k$ and $f = 1$.

Suppose now $e > 0$.

Case 1. G has a cycle.

Delete one edge from a cycle. In the new graph:

$e' = e - 1$, $n' = n$, $f' = f - 1$ (Jordan!), and $k' = k$.

Case 2. G is a forest.

Delete a pendant edge. In the new graph:

$e' = e - 1$, $n' = n$, $f' = f$, and $k' = k + 1$.

Remark. The dual may depend on the embedding of the graph, but the number of faces does *not*.

When is a graph planar?

Corollary If G is a simple, planar graph with $v(G) \geq 3$, then $e(G) \leq 3v(G) - 6$.
If also G is triangle-free, then $e(G) \leq 2v(G) - 4$.

Corollary K_5 and $K_{3,3}$ are non-planar.

The subdivision of edge $e = xy$ is the replacement of e with a new vertex z and two new edges xz and zy. The graph H' is a subdivision of H, if one can obtain H' from H by a series of edge subdivisions. Vertices of H' with degree at least three are called branch vertices.

Theorem (Kuratowski, 1930) A graph G is planar iff G does not contain a subdivision of K_5 or $K_{3,3}$.

Coloring maps with 5 colors

Six Color Theorem. If G is planar, then $\chi(G) \leq 5$.

Proof. By Euler, minimum degree is at most 5. Then

Proposition $\chi(G) \leq \max_{H \subseteq G} \delta(H) + 1$.

Proof. Greedy coloring procedure with the ordering $v_1, \ldots, v_n$, where v_i is a min-degree vertex of the graph $G[\{v_1, \ldots, v_n\}]$.

Five Color Theorem. (Heawood, 1890) If G is planar, then $\chi(G) \leq 5$.

Proof. Take a minimal counterexample.

(i) There is a vertex v of degree at most 5.

(ii) Modify a proper 5-coloring of $G - v$ to obtain a proper 5-coloring of G. A contradiction.
(*Idea of modification:* Kempe chains.)

Coloring maps with 4 colors

Four Color Theorem. (Appel-Haken, 1976) For any planar graph G, $\chi(G) \leq 4$.

Idea of the proof.
W.l.o.g. we can assume G is a planar triangulation.
A configuration in a planar triangulation is a separating cycle C (the ring) together with the portion of the graph inside C.
For the Four Color Problem, a set of configurations is an unavoidable set if a minimum counterexample must contain a member of it.
A configuration is reducible if a planar graph containing it cannot be a minimal counterexample.

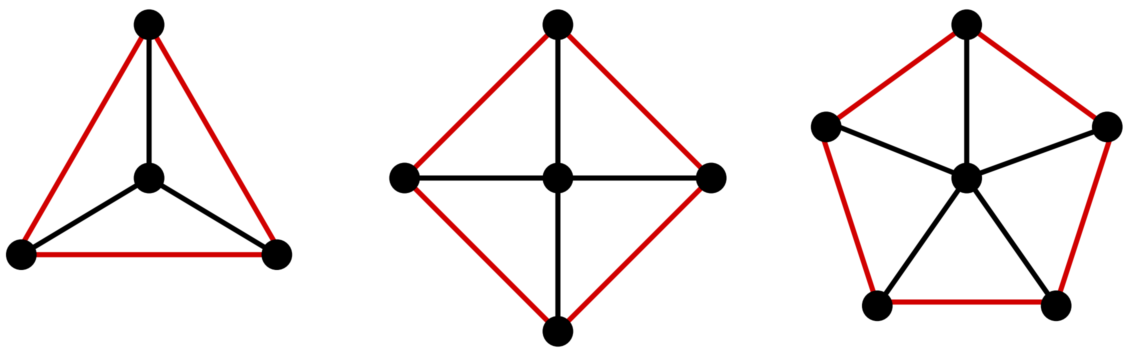
The usual proof attempts to

(i) find a set $\mathcal{C}$ of unavoidable configurations, and

(ii) show that each configuration in $\mathcal{C}$ is reducible.

Proof attempts of the Four Color Theorem

Kempe's original proof tried to show that the unavoidable set



is reducible.

Appel and Haken found an unavoidable set of 1936 of configurations, (all with ring size at most 14) and proved each of them is reducible. (1000 hours of computer time)

Robertson, Sanders, Seymour and Thomas (1996) used an unavoidable set of 633 configuration. They used 32 rules to prove that each of them is reducible. (3 hours computer time)