Connectivity

A vertex cut of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component.

For $G \neq K_n$, the connectivity of G is

$$\kappa(G) := min\{|S| : S \text{ is a vertex cut}\}.$$

By definition, $\kappa(K_n) := n - 1$.

A graph G is k-connected if (1) $v(G) \geq k + 1$ and (2) there is no vertex cut of size $k - 1$. (i.e. $\kappa(G) \geq k$)

Examples. $\kappa(K_{n,m}) = \min\{n, m\}$
$\kappa(Q_d) = d$

Proposition $\delta(G) \leq \kappa(G)$

Characterization of 2-connected graphs

Theorem. G is 2-connected if and only if for every $u, v \in V(G)$, there is a cycle containing both u and v.

Proof. Induction on the distance between u and v. Let w be the penultimate vertex on a shortest path from u to v. Combine the edge vw and a u,v-path P of $G-w$ with two internally disjoint u,w-paths R and Q of G to find two internally disjoint u,v-paths.