Fibonacci sequence

- Leonardo of Pisa, 13th century, rabbits
- comes up often in nature and math/cs

$F_0 = 0 \quad , F_1 = 1$

$\forall n \geq 2 \quad F_n = F_{n-1} + F_{n-2}$

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

Formula ???

Remark: Would be easy if $F_n = F_{n-1} + F_{n-1} \rightsquigarrow 2^n$
or $F_n = F_{n-2} + F_{n-2} \rightsquigarrow \sqrt{2}^n$

$\boxed{\text{Idea}}$

Encode sequence as ONE FUNCTION
(infinitely many numbers)

$$f(x) = \sum_{i=0}^{\infty} F_i x^i = F_0 + F_1 x + F_2 x^2 + \ldots$$

How does this help?

Use recurrence $\rightsquigarrow$ equation for the function

$$x f(x) = \sum_{i=0}^{\infty} F_i x^{i+1} = \qquad F_0 x + F_1 x^2 + F_2 x^3 + F_3 x^4 + \ldots$$

$$+ \quad x^2 f(x) = \sum_{i=0}^{\infty} F_i x^{i+2} = \qquad F_0 x^2 + F_1 x^3 + F_2 x^4 + \ldots$$

$$x f(x) + x^2 f(x) = F_0 x + \sum_{i=2}^{\infty} (F_{i-2} + F_{i-1}) x^i = F_0 x + (F_0 + F_1) x^2 + (F_1 + F_2) x^3 + (F_2 + F_3) x^4 + \ldots$$

compare to

$$f(x) = \sum_{i=0}^{\infty} F_i x^i = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \ldots$$

$$x f(x) + x^2 f(x) - f(x) = \overset{0}{\overset{\|}{F_0}} x - \overset{0}{\overset{\|}{F_0}} - \overset{x}{\overset{\|}{F_1 x}}$$

$$f(x) = \frac{x}{1 - x - x^2}$$

- Find series expansion (Taylor series)

  - Partial fractions: ~~xxxxx~~ $x^2 - x - 1 = 0$

$$x_{1,2} = \frac{+1 \pm \sqrt{5}}{2} \qquad \alpha_1 = \frac{-\sqrt{5}+1}{2} \quad \alpha_2 = \frac{\sqrt{5}+1}{2}$$

$$\frac{x}{1-x-x^2} = \frac{A}{1-\alpha_1 x} + \frac{B}{1-\alpha_2 x}$$

$$= \frac{x(-\alpha_2 A - \alpha_1 B) + A + B}{(1-\alpha_1 x)(1-\alpha_2 x)}$$

$$(1-\alpha_1 x)(1-\alpha_2 x) = 1 - (\alpha_1+\alpha_2)x + \alpha_1\alpha_2 x^2 = 1 - x - x^2$$

$$\Longrightarrow -\alpha_2 A - \alpha_1 B = 1$$

$$A + B = 0$$

$$\Longrightarrow -\alpha_2 A + \alpha_1 A = 1 \implies A = \frac{1}{\alpha_1 - \alpha_2} = -\frac{1}{\sqrt{5}}$$

$$B = \frac{1}{\sqrt{5}}$$

  - Express as power series (⇝ here: geometric series)

$$\frac{1}{1-\alpha_1 x} = \sum_{i=0}^{\infty} (\alpha_1 x)^i \qquad \frac{1}{1-\alpha_2 x} = \sum_{i=0}^{\infty} (\alpha_2 x)^i$$

$$\sum_{i=0}^{\infty} F_i x^i = f(x) = \frac{x}{1-x-x^2} = A\sum_{i=0}^{\infty}(\alpha_1 x)^i + B\sum_{i=0}^{\infty}(\alpha_2 x)^i = \sum_{i=0}^{\infty}(A\alpha_1^i + B\alpha_2^i)\, x^i$$

$$\Longrightarrow \forall i \in \mathbb{N}\; F_i = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^i - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^i$$

$$F_i = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i\right)$$

- Is this an integer ???

YES (we have proved it that it is the Fibonacci number

- How big?

$$\frac{1}{\sqrt{5}}\left|\left(\frac{1-\sqrt{5}}{2}\right)^i\right| < \frac{1}{2} \Rightarrow$$

$$\Rightarrow F_i = \text{integer closest to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i$$

Generating functions

$\boxed{\text{Idea}}$

Encode a sequence as a power series
(many numbers $\leftrightsquigarrow$ one function)

$$(a_0, a_1, a_2, \dots) \longleftrightarrow a_0 + a_1 x + a_2 x^2 + \dots = \sum_{i=0}^{\infty} a_i x^i$$

Example:

$$(1, 1, 1, \dots, 1, \dots) \longleftrightarrow \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

convergent for $|x| < 1$

Proposition

Let $(a_0, a_1, a_2, \dots)$ be a sequence of reals.
If $|a_n| < K^n \ \forall n \in \mathbb{N} \implies \forall x \in \left(-\frac{1}{K}, \frac{1}{K}\right)$ $a(x) = \sum_{i=0}^{\infty} a_i x^i$ is absolute convergent.

$a(x)$ has derivatives of all order at 0 and for all $n \geq 0$ $\quad a_n = \dfrac{a^{(n)}(0)}{n!}$

In particular, the values of $a(x)$ in an arbitrary small neighborhood of 0 determine uniquely the sequence $(a_0, a_1, a_2, \dots)$

Remark: Theory of formal power series

- ANYWAYS: usually result (obtained by illegal methods) can be easily verified by induction

Def: $(a_0, a_1, a_2, \ldots)$ sequence of real numbers. The <u>generating function</u> of this sequence is the power series $a(x) = \sum_{i=0}^{\infty} a_i x^i$

Important examples:

$$\left(0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots, \tfrac{1}{i}, \ldots\right) \leftrightsquigarrow x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots = -\ln(1-x)$$

(convergent for $\forall x \in (-1, 1)$)

$$\left(1, 1, \tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{24}, \ldots, \tfrac{1}{i!}, \ldots\right) \leftrightsquigarrow \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$$

(convergent for $\forall x \in \mathbb{R}$)

$\boxed{r \in \mathbb{R}\ !!!}$

$$\left(\binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \binom{r}{3}, \ldots, \binom{r}{i}, \ldots\right) \leftrightsquigarrow \sum_{i=0}^{\infty} \binom{r}{i} x^i = (1+x)^r$$

(convergent for $\forall x \in (-1, 1)$)

Special Case

$$\frac{1}{(1-x)^n} = (1-x)^{-n} = \sum_{i=0}^{\infty} \binom{-n}{i} (-1)^i x^i = \sum_{i=0}^{\infty} \binom{n-1+i}{n-1} x^i$$

Operations with sequences and their generating fns.

Let $(a_0, a_1, \dots)$ and $(b_0, b_1, \dots)$ be sequences, $\alpha \in \mathbb{C}$

(A) Addition

$$(a_0+b_0, a_1+b_1, \dots) \leftrightsquigarrow a(x) + b(x)$$

(B) Multiplication with a constant

$$(\alpha a_0, \alpha a_1, \dots) \leftrightsquigarrow \alpha a(x)$$

(C) Shifting to the right

$$(\underbrace{0, 0, \dots, 0}_{n}, a_0, a_1, \dots) \leftrightsquigarrow x^n a(x)$$

(D) Shifting to the left

$$(a_n, a_{n+1}, a_{n+2}, \dots) \leftrightsquigarrow \frac{a(x) - a_0 - a_1 x - \dots - a_{n-1}x^{n-1}}{x^n}$$

(E) Substitute αx for x

$$(a_0, \alpha a_1, \alpha^2 a_2, \alpha^3 a_3, \dots) \leftrightsquigarrow a(\alpha x)$$

Example: $(1, 2, 4, 8, \dots, 2^n, \dots) \leftrightsquigarrow$ $(1, 1, \dots)$ $a(2x) = \dfrac{1}{1-2x}$

Application: Solving linear homogeneous recurrences with constant coefficients

Given: • $k \in \mathbb{N}$

• $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in \mathbb{R}$

• sequence $(a_0, a_1, a_2, \ldots, a_i, \ldots)$

with recurrence $a_n = \alpha_{k-1} a_{n-1} + \alpha_{k-2} a_{n-2} + \ldots + \alpha_1 a_{n-k+1} + \alpha_0 a_{n-k}$

$\boxed{\forall n \geq k}$

(linear, homogenous, with constant coefficients)

Using generating functions to solve

$$a(x) = \sum_{n=0}^{\infty} a_n x^n \qquad\qquad +$$

$$x a(x) = \sum_{n=1}^{\infty} a_{n-1} x^n \qquad \Big| \cdot \alpha_{k-1} \qquad -$$

$$x^2 a(x) = \sum_{n=2}^{\infty} a_{n-2} x^n \qquad \Big| \cdot \alpha_{k-2} \qquad -$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$x^k a(x) = \sum_{n=k}^{\infty} a_{n-k} x^n \qquad \Big| \cdot \alpha_0 \qquad -$$

$$a(x) - \alpha_{k-1} x a(x) - \alpha_{k-2} x^2 a(x) - \ldots - x^k a(x) = \begin{array}{l} a_0 + a_1 x + a_2 x^2 + \ldots + a_{k-1} x^{k-1} \\ - \alpha_{k-1}(a_0 x + a_1 x^2 + \ldots + a_{k-2} x^{k-1}) \\ - \alpha_{k-2}(a_0 x^2 + \ldots + a_{k-3} x^{k-1}) \\ \vdots \\ - \alpha_0 a_0 x^{k-1} \end{array}$$

$$R(x) =$$

with $\deg R(x) \leq k-1$

$$a(x) = \frac{R(x)}{1 - \alpha_{k-1} x - \alpha_{k-2} x^2 - \ldots - \alpha_0 x^k}$$

There is partial fraction decomposititon

$$a(x) = \frac{A_1}{1-\lambda_1 x} + \dots + \frac{A_k}{1-\lambda_k x} = A_1 \sum (\lambda_1 x)^i + \dots + A_k \sum (\lambda_k x)^i$$

$$= \sum_{i=0}^{\infty} \left(A_1 \lambda_1^i + \dots + A_k \lambda_k^i\right) x^i$$

provided the characteristic polynomial of the recurrence $p(z) = z^k - \alpha_{k-1} z^{k-1} - \dots - \alpha_1 z - \alpha_0$

$$= (z-\lambda_1) \cdot \ldots \cdot (z-\lambda_k)$$

has k $\boxed{\text{distinct}}$ roots $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{C}$.

[Then $(1-\lambda_1 x) \dots (1-\lambda_k x) = 1 - \alpha_{k-1} x - \dots - \alpha_0 x^k$

$\underset{z=\frac{1}{x}}{\Longleftrightarrow} \quad (z-\lambda_1) \dots (z-\lambda_k) = z^k - \alpha_{k-1} z^{k-1} - \dots - \alpha_0$]

(Assume $\lambda_i \neq 0$, as then $\alpha_0 = 0 \Rightarrow$ recurrence is NOT degree k)

Initial conditions $(a_0, a_1, \dots, a_{k-1})$ and coefficients $(\alpha_0, \alpha_1, \dots, \alpha_{k-1})$ determine $A_1, \dots, A_k \in \mathbb{C}$

and formula $\underline{a_n = A_1 \lambda_1^n + \dots + A_k \lambda_k^n} \quad \underline{\forall n \geq 0}$

Theorem: let • $k \in \mathbb{N}$
• $\alpha_0, \dots, \alpha_{k-1} \in \mathbb{R}$
• sequence with recurrence

$$(*)\quad a_n = \alpha_{k-1} a_{n-1} + \dots + \alpha_0 a_{n-k} \quad \boxed{\forall n \geq k}$$

• characteristic polynomial

$$p(x) = x^k - \alpha_{k-1} x^{k-1} - \dots - \alpha_0$$
$$= (x-\lambda_1) \cdot \dots \cdot (x-\lambda_k) \text{ with roots } \lambda_1, \dots, \lambda_k \in \mathbb{C}$$

(i) (Simple Roots) Suppose $\lambda_i \neq \lambda_j \quad \forall i \neq j$.

Then for ANY sequence $(a_0, a_1, a_2, \dots)$ satisfying $(*)$

$$\exists C_1, \dots, C_k \in \mathbb{C} \text{ s.t. } \boxed{\forall n \geq 0} \; C_1 \lambda_1^n + \dots + C_k \lambda_k^n = a_n$$

(ii) (General Case) $\boxed{\text{HW}}$

Suppose $p(x) = (x-\lambda_1)^{k_1} \cdot \dots \cdot (x-\lambda_q)^{k_q}$ with $k_1 + \dots + k_q = k$ and $\lambda_i \neq \lambda_j$ for $i \neq j$

The for ANY sequence $(a_0, a_1, \dots)$ satisfying $(*)$

$$\exists C_{ij} \begin{pmatrix} \forall i = 1, \dots, q \\ j = 0, \dots, k_i - 1 \end{pmatrix} \text{ s.t. } \boxed{\forall n \geq 0} \; a_n = \sum_{i=1}^{q} \sum_{j=0}^{k_i - 1} C_{ij} n^j \lambda_i^n$$

(F) Substituting x^n for x

$$(a_0, \underbrace{0, \dots, 0}_{n-1}, a_1, \underbrace{0, \dots, 0}_{n-1}, a_2, 0 \dots) \rightsquigarrow a(x^n) = \sum_{i=0}^{\infty} a_i x^{ni}$$

Example: Generating fn of $a_n = 2^{\lfloor \frac{n}{2} \rfloor}$

That is $(1, 1, 2, 2, 4, 4, 8, 8, \dots)$

Pf: $(1, 1, 1, \dots) \rightsquigarrow \dfrac{1}{1-x}$

Substitute $2x$

$(1, 2, 4, 8, \dots) \rightsquigarrow \dfrac{1}{1-2x}$

Substitute x^2

$(1, 0, 2, 0, 4, 0, 8, \dots) \rightsquigarrow \dfrac{1}{1-2x^2}$

Shift to right by 1

$(0, 1, 0, 2, 0, 4, 0, 8, \dots) \rightsquigarrow \dfrac{x}{1-2x^2}$

$+ \; = (1, 1, 2, 2, 4, 4, 8, 8, \dots) \rightsquigarrow \dfrac{1+x}{1-2x^2}$

(G) Differentiation / Integration

$a'(x) \longleftrightarrow (a_1, 2a_2, 3a_3, \ldots, na_n, \ldots)$

$\int_0^x a(t)\,dt \longleftrightarrow \left(0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2, \ldots\right)$

Example: generating fn of $a_n = (n+1)^2$

That is of $(1^2, 2^2, 3^2, \ldots)$

$(1, 1, 1, \ldots) \longleftrightarrow \frac{1}{1-x}$

Differentiate

$(1, 2, 3, 4, \ldots) \longleftrightarrow \frac{1}{(1-x)^2}$

Differentiate

$(1\cdot 2, 2\cdot 3, 3\cdot 4, \ldots) \longleftrightarrow \frac{2(1-x)}{(1-x)^4} = \frac{2}{(1-x)^3}$

Subtract

$\bigcirc - \bigcirc = (1^2, 2^2, 3^2, \ldots) \longleftrightarrow \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}$

H) Product of generating fns

$$a(x)\cdot b(x) \longleftrightarrow (c_0, c_1, c_2, \ldots\ldots)$$

where $c_0 = a_0 b_0$

$c_1 = a_0 b_1 + b_1 a_0$

$\vdots$

$c_n = \sum_{k=0}^{n} a_k b_{n-k}$

<u>Proposition</u>: ~~Let~~ $a_n :=$ # of ways to build a structure of Type I on ~~any set~~ $[n]$

$b_n :=$ # of ways to build a structure of Type II on $[n]$

$c_n :=$ # of ways to first separate $[n]$ into two intervals $I = [i]$ and $[n] \setminus I$ and then build a structure of Type I on I and a structure of Type II on $[n] \setminus I$.

Then for the generating fns.

$$C(x) = A(x) \cdot B(x)$$

<u>Pf</u>: $\forall\, i = 0, 1, \ldots, n$ $\exists$ a_i ways to build on $[i]$

and b_{n-i} — " — $[n] - [i]$

$\Rightarrow a_i b_{n-i}$ to build on both

$$\Rightarrow c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

Example: Design term in an Engineering Department

Term : n days

Decisions: • How long should theoretical part be?

k days $(1 \le k \le n-2)$

Rest is laboratory part ($n-k$ days)

• When should ~~the~~ be the

× project day for the theoretical part?

× two project days for the lab part?

How many ways are there? $\boxed{f_n}$

Product Formula:

$a_n :=$ # ways to select project day from an n day long theoretical part

$a_n = n$

$b_n :=$ # ways to select TWO project days from an n day long lab part

$b_n = \binom{n}{2}$

$$a(x) = \sum_{n=0}^{\infty} n x^n = x \cdot \left(\sum_{n=0} x^n\right)' = x \cdot \left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}$$

$$b(x) = \sum_{n=2}^{\infty} \binom{n}{2} x^n = \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} x^n\right)'' = x^2 \left(\frac{1}{1-x}\right)'' = \frac{x^2}{(1-x)^3}$$

$$f(x) = \sum f_n x^n = a(x) \cdot b(x) = \frac{x^3}{(1-x)^5} = x^3 \sum_{n=0}^{\infty} \binom{n+4}{4} x^n = \sum_{n=3}^{\infty} \binom{n+1}{4} x^n$$

$\longrightarrow f_n = \binom{n+1}{}$