Möbius function of a poset

$\boxed{\text{Example 0.}}$

$a_0, a_1, a_2, \ldots \in \mathbb{R} \longrightarrow (\underset{b_0}{\underset{\shortparallel}{a_0}}, \underset{b_1}{\underset{\shortparallel}{a_0+a_1}}, \ldots, \underset{b_n}{\underset{\shortparallel}{\sum_{i=0}^{n} a_i}}, \ldots)$

$\boxed{\text{Then}}$

$\forall n \geq 1 \quad \boxed{a_n = b_n - b_{n-1}}$

backwards formula

$\boxed{\text{Example 1.}}$

Let $f: 2^{[n]} \longrightarrow \mathbb{R} \rightsquigarrow$ Define $g: 2^{[n]} \longrightarrow \mathbb{R}$ by

$$g(T) = \sum_{S \subseteq T} f(S)$$

Then backwards

Thm: $\forall T \in 2^{[n]} \quad f(T) = \sum_{S \subseteq T} g(S)(-1)^{|T \setminus S|}$

Pf: $\sum_{S \subseteq T} g(S)(-1)^{|T\setminus S|} = \sum_{S \subseteq T} (-1)^{|T \setminus S|} \sum_{S' \subseteq S} f(S') =$

Def of g

$$= \sum_{S' \subseteq T} f(S') \sum_{\substack{S \\ S' \subseteq S \subseteq T}} (-1)^{|T \setminus S|} = f(T)$$

Only term that remains is when $k = |T \setminus S'| = 0$

$\shortparallel$ Let $k = |T \setminus S'|$

$$(-1)^{k} + \binom{k}{1}(-1)^{k-1} + \binom{k}{2}(-1)^{k-2} + \ldots + \binom{k}{k}(-1)^0 = (1-1)^k = \begin{cases} 0 & k \geq 1 \\ 1 & k = 0 \end{cases}$$

$\boxed{\text{Example 2}}$

• Let $f: \mathbb{N} \to \mathbb{R}$ $\rightsquigarrow$ Define $g(n) := \sum_{d|n} f(d)$

backwards?

$f(n) =$?? in terms of g ??

Anything nice?

Common generalization: functions on posets

Example 0	Example 1.	Example 2.
$(a_i)_{i\in\mathbb{N}_0}$ function on $(\mathbb{N}_0, \leq)$	function on $(2^{[n]}, \subseteq)$	function on $(\mathbb{N}, \mid)$

Function $f: P \longrightarrow \mathbb{R}$ where $(P, \leq)$ is a poset

Define $g(y) := \sum_{x \leq y} f(x) \quad \forall y \in P$

Sum up values of f for every element smaller or equal to y in P

Question

$$\boxed{\text{What is } f \text{ in terms of } g\text{?}}$$

~~Def~~ poset $(P,<)$ is locally finite

if $\forall a,b \in P$ the interval $[a,b] := \{x \in P : a \leq x \leq b\}$ is finite

Examples: • $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$
• $(2^S, \subseteq)$
• $(\mathbb{N}, \cdot|\cdot) \longrightarrow$ divisibility

Def: incidence algebra $I(P)$ of P (over $\mathbb{R}$)

is the set of fns. $\{f : P^2 \longrightarrow \mathbb{R} : f(x,y) = 0 \text{ if } x \not\leq y\}$

with operations: addition: $(f+g)(x,y) = f(x,y) + g(x,y)$

multiplication with constant: $(\lambda f)(x,y) = \lambda f(x,y)$

multiplication: $(f * g)(a,b) = \sum_{a \leq x \leq b} f(a,x) \cdot g(x,b)$

• $I(P)$ is set of function on intervals (whenever $x \not\leq y$, so interval is empty, $f(x,y) = 0$)
• $*$ is well-defined (sum is finite $\Leftarrow$ local finiteness)
• $(f*g)(a,b) = 0$ if $a \not\leq b$ (empty sum)
• $*$ is associative (matrix multiplication of upper triangular matrices (rows/columns in an order according to arbitrary linear extension of P)

- $\exists$ unit element (both right and left-sided)

  Kronecker-delta $\delta(x,y) := \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x\neq y \end{cases}$ ("unit matrix")

  Verify!

- $f \in I(P)$ has a (unique) (right AND left sided) inverse

  $\Updownarrow$

  $f(x,x) \neq 0 \quad \forall x \in P$ (upper triangular matrix is invertible $\Leftrightarrow$ $\forall$ digonal entry is $\neq 0$)

[One can calculate it recursively:

$$f^{-1}(x,x) := f(x,x)^{-1}$$

Once $f^{-1}(x,z)$ is defined for $\forall z \in [x,y] \setminus \{y\}$

we have
$$f^{-1}(x,y) = \frac{1}{f(y,y)}\left(-\sum_{\substack{z \\ x\leq z<y}} f^{-1}(x,z)\, f(z,y)\right)$$

OR if $f^{-1}(z,y)$ is defined for $\forall z \in [x,y] \setminus \{x\}$

we have
$$f^{-1}(x,y) = \frac{1}{f(x,x)}\left(-\sum_{\substack{z \\ x<z\leq y}} f^{-1}(x,z)\, f(z,y)\right)$$
]

Def: $\zeta_P \in I(P)$ is the zeta-function of P defined by $\zeta_P(x,y) := \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{if } x \not\le y \end{cases}$

(characteristic fn of the relation $\le$)

Proposition: P is a locally finite poset

$\forall x, y \in P$ $\quad (\zeta - \delta)^k(x,y) = \#$ of x,y-chains of length k

Pf: By induction on k.

$[k = 0 \quad \rightsquigarrow \quad (\zeta - \delta)^0(x,y) = \delta(x,y)]$

$k = 1 \quad \rightsquigarrow \quad (\zeta - \delta)^1(x,y) = 1 \iff \exists$ an x,y chain of length 1

Let $k > 1 \quad x, y \in P$.

Classify ~~all~~ chains $x = x_0 \lneq x_1 \lneq x_2 \cdots \lneq x_{k-1} \lneq x_k = y$

according to next-to last element x_{k-1}

$$\#\text{of } x,y\text{-chains of length } k = \sum_{x \le z \le y} \underbrace{(\zeta-\delta)^{k-1}(x,z)}_{\substack{\| \\ \#\text{ of } x,z\text{-chains} \\ \text{of length } k-1}} \underbrace{(\zeta-\delta)(z,y)}_{\substack{\| \\ \#\text{ of } (z,y)\text{-chains} \\ \text{of length } 1}}$$

$$(\zeta-\delta)^{k-1} * (\zeta-\delta)(x,y) = (\zeta-\delta)^k(x,y)$$

$\square$

Def: $\mu_P = \zeta_P^{-1} \in I(P)$ is called the Möbius function of P

Prop: $\mu = \mu_P$ exists: $\forall a \in P \quad \mu(a,a) = 1$

$$\forall a \lneq b \quad \mu(a,b) = -\sum_{a \le z < b} \mu(a,z) = -\sum_{\substack{z \\ a < z \le b}} \mu(z,b)$$

Pf:

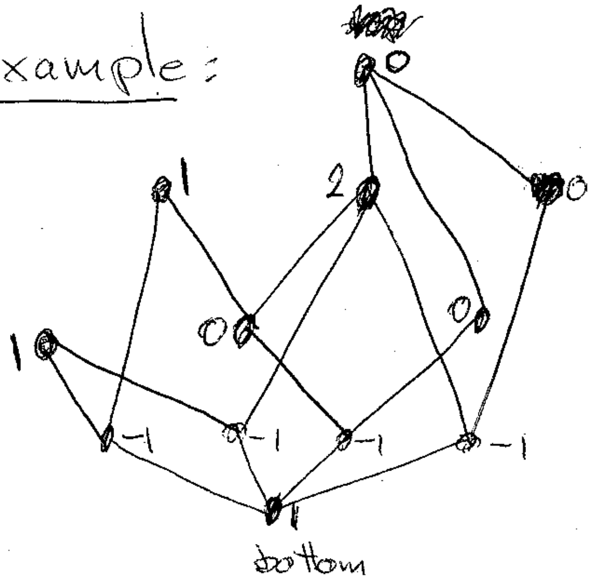
$$1 = \delta(a,a) = \mu * \zeta(a,a) = \mu(a,a) \cdot \overset{1}{\overset{\parallel}{\zeta(a,a)}} = \mu(a,a)$$

$$a \lneq b \quad \underline{\underline{0}} = \mu * \zeta(a,b) = \sum \mu(a,z)\, \overset{1}{\overset{\parallel}{\zeta(z,b)}} = \sum_{\substack{z \\ a \le z \le b}} \mu(a,z) = \sum_{\substack{z \\ a \le z < b}} \mu(a,z) + \mu(a,b)$$

$$\underline{\underline{0}} = \zeta * \mu(a,b) = \sum \zeta(a,z) \cdot \mu(z,b) = \sum_{\substack{z \\ a \le z \le b}} \mu(z,b) = \sum_{\substack{z \\ a < z \le b}} \mu(z,b) + \mu(a,b)$$

$\square$

Example:



$\mu(\text{bottom}, x) = ?$

Examples: • $(\mathbb{N}_0, \le)$ $\quad \mu(x,y) = \begin{cases} 1 & x = y \\ -1 & y = x+1 \\ 0 & \text{otherwise} \end{cases}$

Pf: Induction on $|y-x|$

• $(2^{[n]}, \subseteq)$ $\quad \mu(S,T) = (-1)^{|T \setminus S|}$

Pf: Induction on $|T \setminus S|$

• $(\mathbb{N}, \cdot|\cdot)$ $\quad \mu(x,y) = \begin{cases} (-1)^k & \text{if } \frac{y}{x} = p_1 \cdot p_2 \cdots p_k \text{ where } p_1, \dots, p_k \text{ are all distinct} \\ 0 & \text{if } \frac{y}{x} \text{ is divisible by a square} \end{cases}$

Pf: (1) $([x,y], \cdot|\cdot) \cong ([1, \tfrac{y}{x}], \cdot|\cdot)$ poset isomorphism

$z \longleftrightarrow \frac{z}{x}$

(2) if $y = p_1 \cdots p_k$ with distinct primes $p_1, \dots, p_k$

then $([1,y], \cdot|\cdot) \cong (2^{[k]}, \subseteq)$

$S \subseteq [k] \quad \prod_{i \in S} p_i = z \longleftrightarrow S$

$\Longrightarrow \mu\left(\prod_{i=1}^{k} p_i\right) = (-1)^k$

(3) If y is divided by a square $\rightsquigarrow$ induction

$$\mu(y) = -\sum_{\substack{z \mid y \\ z \ne y}} \mu(z) = -\sum_{\substack{z \mid y \\ z \text{ is squarefree}}} \mu(z) - \sum_{\substack{z \mid y \\ z \ne y \\ z \text{ is NOT squarefree}}} \underbrace{\mu(z)}_{0 \text{ by induction}} = 0$$

$$-\sum_{\substack{z \mid y \\ z \text{ is squarefree}}} \mu(z) = -\sum_{i=0}^{k} (-1)^i \binom{k}{i}$$

Thm: (Möbius Inversion Formula)

Let P be locally finite poset.

Let $f, \cancel{g} : P \longrightarrow \mathbb{R}$. ~~Define $g(y) = \sum_{x \leq y} f(x)$~~

(i) (Inversion from below) $\boxed{\text{Suppose } \exists \text{ a minimum element } 0 \in P}$

If $g(y) = \sum_{x \leq y} f(x)$ $(\forall y)$ Then $(\forall y)$ $f(y) = \sum_{x \leq y} g(x)\, \mu(x,y)$

(ii) (Inversion from above) $\boxed{\text{Suppose } \exists \text{ a maximum element } 1 \in P}$

If $g(y) = \sum_{x \geq y} f(x)$ $(\forall y)$ Then $f(y) = \sum_{x \geq y} g(x)\, \mu(y,x)$ $(\forall y)$

Pf: Order coordinates according to arbitrary lin extension

(i) $M = \begin{array}{cc} & y \\ x & \begin{pmatrix} 1 & & \mu(x,y) \\ & 1 & \\ 0 & & \ddots \end{pmatrix} \end{array}$ $\qquad Z = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$

$\vec{f} = (f(0), \dots \underset{\substack{\uparrow \\ x}}{f(x)}, \dots)$ $\qquad \vec{g} = (g(0), \dots \underset{\substack{\uparrow \\ x}}{g(x)}, \dots)$

$\forall y\ g(y) = \sum_{x \leq y} f(x) \iff \vec{g} Z = \vec{f} \iff \vec{g} Z M = \vec{f} M \iff \vec{g} = \vec{f} M$

$\iff g(y) = \sum_{x \leq y} g(x)\, \mu(x,y)$

Application:

Number theoretic Möbius Inversion

Def: Möbius function $\mu : \mathbb{N} \to \{-1, 1, 0\}$

$$\mu(n) := \begin{cases} 1 & n \text{ is the product of an } \underline{\text{even}} \text{ \# of distinct primes} \\ -1 & \text{ -- " -- } \quad \underline{\text{odd}} \text{ -- " --} \\ 0 & \text{otherwise} \end{cases}$$

Example: $\mu(30) = -1 \qquad \mu(77) = 1 \qquad \mu(24) = 0$

$\boxed{\text{(Number theoretic) Möbius Inversion}}$ Formula

Let $f(n) : \mathbb{N} \to \mathbb{C}$. If function $g : \mathbb{N} \to \mathbb{C}$ is defined by $g(n) := \sum_{d \mid n} f(d) \implies f(n) = \sum_{d \mid n} g(d) \mu\left(\frac{n}{d}\right)$

$\forall n$