# Examples of Möbius Inversion

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## Introduction

Exciting! Mysterious! Compelling! These are all the things I was taught an introduction should be, and these are all things this introduction is not. For, you see, this note is not a typical note intended for independent consumption, but rather a continuation[1] of the Discrete Maths I lecture delivered on June 8th, 2016. As such, I shall assume that you, the reader, are not some generic Internetter browsing the World Wide Web at random, whose attention I must grab before you depart on your search for online entertainment,[4] but instead a student who, fresh from your long overdue break, are ready to learn more about Möbius inversion. This introduction is therefore intended to be a brief review of what we covered in lecture, reminding you of the relevant results (but not their proofs[5]).

The topic of the day had been Möbius inversion, a process for inverting summation over posets. The general result we obtained is reproduced below.

**Theorem 1.** *Let $(P, \leq)$ be a finite[6] poset, and suppose we have a function $f : P \to \mathbb{R}$. If $g : P \to \mathbb{R}$ is defined by*

$$g(y) = \sum_{x \leq y} f(x),$$

*then*

$$f(y) = \sum_{x \leq y} g(x) \mu_{x,y},$$

*where $\mu$, the Möbius function of the poset, is given by*

$$\mu_{x,y} = \begin{cases} 0 & \text{if } x \not\leq y, \\ 1 & \text{if } x = y, \text{ and} \\ -\sum_{x \leq z < y} \mu_{x,z} & \text{if } x < y. \end{cases} \tag{1}$$

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[1]Were I a braver man, I would perhaps have remained at the blackboard for another forty-five minutes or so, until I had finished presenting this material. However, I instead yielded to both internal[2] and external[3] forces, and resolved to conclude the lecture through this electronic medium.

[2]Deprived of their customary break for a second consecutive day, my audience was rustling restlessly and eager to see the lecture end.

[3]The next class were waiting in the wings to take over the classroom, and I did not want to test their patience any further than I already had.

[4]Have you tried `http://littleanimalgifs.tumblr.com/`?

[5]I think I will one day return to fill in these missing details, so that this note may be self-contained, but I also thought I would go to the gym today. Sadly, my best intentions often remain unfulfilled.

[6]The finiteness of the poset is not strictly needed, but something we assumed for convenience. The result holds more generally for *locally finite* posets, which are posets where every interval $[x, y] = \{z \in P : x \leq z \leq y\}$ is finite,[7] provided we take $g$ and $f$ to be sums over some interval: for some fixed $x_0 \in P$, $g(y) = \sum_{x_0 \leq x \leq y} f(x)$, and then $f(y) = \sum_{x_0 \leq x \leq y} g(x) \mu_{x,y}$.

[7]This local finiteness is necessary, for otherwise we would be dealing with infinite sums and our expressions would not be well-defined.

One very important feature of Möbius inversion is that the Möbius function $\mu$ depends only on the poset $(P, \leq)$, and not on the functions $f$ or $g$. Hence, when we wish to carry out Möbius inversion on some given poset, we only need to compute the Möbius function once, and can then use it for any and all functions on the poset. It is thus worth our time to compute the Möbius function for some standard posets, which we shall do in the sequel.

## The naturals with their usual order

We begin with an example where the inversion procedure is obvious, and all the machinery of Möbius inversion seems unnecessary. We do this for two reasons: to demonstrate the versatility of the method, to show that it works in the simplest of domains as well as the most complicated, as well as to make it easy for you to confirm that our formula in Theorem 1 may indeed be correct.

The poset in question will be the set $[n]$ of the first $n$ natural numbers, together with the (total) order given by the usual order $\leq$.[8]

**Proposition 2.** *For the poset $([n], \leq)$, the Möbius function is*

$$\mu_{x,y} = \begin{cases} 1 & \text{if } y = x, \\ -1 & \text{if } y = x+1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We prove the proposition by induction on $y - x$.

For the base cases, if $y - x < 0$, then $x \not\leq y$, and hence by (1) we must have $\mu_{x,y} = 0$, as claimed. If $y - x = 0$, then by (1) we again must have $\mu_{x,x} = 1$, as given. Finally, if $y - x = 1$, then (1) gives $\mu_{x,x+1} = -\sum_{x \leq z < x+1} \mu_{x,z} = -\mu_{x,x} = -1$.

For the induction step, suppose $y - x \geq 2$. Using (1) and the induction hypothesis, we find

$$\mu_{x,y} = -\sum_{x \leq z < y} \mu_{x,z} = -\left(\mu_{x,x} + \mu_{x,x+1} + \sum_{x+2 \leq z < y} \mu_{x,y}\right) = -\left(1 + (-1)\right) = 0,$$

completing the induction step. □

Now that we know the Möbius function, we can perform inversion over this poset.

**Corollary 3.** *If we have a function $f : \mathbb{N} \to \mathbb{R}$, and $g : \mathbb{N} \to \mathbb{R}$ is given by $g(n) = \sum_{i=1}^{n} f(i)$, then $f(n) = g(n) - g(n-1)$.*

*Proof.* For fixed $n \in \mathbb{N}$, we consider these functions restricted to $[n]$. By Theorem 1 and Proposition 2,

$$f(n) = \sum_{i=1}^{n} g(i)\mu_{i,n} = g(n) - g(n-1).$$ □

[8]If you are comfortable with the extension of Theorem 1 to locally finite posets, then you will see that this result extends to the poset $(\mathbb{N}, \leq)$.

Note that the sequence $(g(n))_{n\in\mathbb{N}}$ represents the partial sums of the sequence $(f(n))_{n\in\mathbb{N}}$, and I have no doubt that throughout your mathematical career, you have encountered partial sums in several settings. You therefore need no convincing that being able to invert partial sums is a useful task, but also know that it is trivial, and can confirm that Corollary 3 gives the right answer.

## The Boolean poset

In lecture we also considered the case of the Boolean poset, studied by combinators and discrete geometers and theoretical computer scientists alike. Here the poset is $(P, \leq) = (2^{[n]}, \subseteq)$; that is, all subsets of the first $n$ natural numbers, ordered by inclusion. Using induction, we determined the Möbius function, as given below.

**Proposition 4.** *For the poset $(2^{[n]}, \subseteq)$, the Möbius function is*

$$\mu_{S,T} = \begin{cases} (-1)^{|T \setminus S|} & \text{if } S \subseteq T, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Again, by Theorem 1, this immediately gives us the formula for Möbius inversion over this poset.

**Corollary 5.** *If we have a function $f : 2^{[n]} \to \mathbb{R}$, and $g : 2^{[n]} \to \mathbb{R}$ is given by $g(T) = \sum_{S \subseteq T} f(S)$, then $f(T) = \sum_{S \subseteq T} (-1)^{|T \setminus S|} g(S)$.*

While you take in this fact, nodding appreciatively, part of you might wonder if anyone would ever really compute such sums over the Boolean poset. To show you that this is not just knowledge, but is useful knowledge, we shall show that the inclusion-exclusion principle is in fact just a special case of Möbius inversion. To state the result more concisely, given sets $A_1, \ldots, A_n \subseteq X$, and some subset of indices $I \subseteq [n]$, we write $A_I = \cap_{i \in I} A_i$ for the intersection of the sets with indices in $I$, and $(A^c)_I = \cap_{i \in I} A_i^c$ for the intersection of their complements. Note that we take $A_\emptyset = X$.

**Corollary 6** (The inclusion-exclusion principle[9]). *Let $A_1, A_2, \ldots, A_n$ be subsets of some finite set $X$. Then $\left|(A^c)_{[n]}\right| = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$.*

*Proof.* To prove the inclusion-exclusion principle, we use Corollary 5 with a clever[10] choice of function $f$. One thing to note is that we work in the poset $(2^{[n]}, \subseteq)$, as we will be concerned with subsets of indices. Our poset does not 'see' the ground set $X$ and the subsets themselves.

Given $S \subseteq [n]$, define $f(S) = \left|(A^c)_S \cap A_{[n] \setminus S}\right|$. This counts the number of elements that are:

(i) missing from all the sets $A_i$ whose indices are in $S$, and

[9] This version of the inclusion-exclusion principle looks a little different from what we had seen earlier in lecture, because there we gave a formula for the size of the union $\cup_{i=1}^n A_i$. You can get the formula below by taking a complement.

[10] This is not my proof, so I'm allowed to say this.

(ii) present in all the sets $A_i$ whose indices are not in $S$.

Note that as $S$ varies over $2^{[n]}$, the sets $(A^c)_S \cap A_{[n]\setminus S}$ partition $X$, since for every $x \in X$, there is a unique $S \subseteq [n]$ such that $x \notin A_i \Leftrightarrow i \in S$.

Now if $g(T) = \sum_{S \subseteq T} f(S)$, $g(T)$ counts the number of elements that are missing from some sets whose indices lie in $T$, but not from any others. That is, $g(T)$ counts the number of elements that are in every set $A_i$ for which $i \notin T$, and hence $g(T) = \left|A_{[n]\setminus T}\right|$.

By Corollary 5,

$$\begin{aligned} f([n]) &= \left|(A^c)_{[n]} \cap A_{[n]\setminus[n]}\right| = \left|(A^c)_{[n]}\right| \\ &= \sum_{S \subseteq [n]} (-1)^{|[n]\setminus S|} g(S) = \sum_{S \subseteq [n]} (-1)^{|[n]\setminus S|} \left|A_{[n]\setminus S}\right| \\ &= \sum_{I \subseteq [n]} (-1)^{|I|} \left|A_I\right|, \end{aligned}$$

where we make the substitution $I = [n] \setminus S$ in the last equality. This proves the inclusion-exclusion principle. □

## The divisibility poset

Our final example leads us to the wonderful world of number theory. In the divisibility poset, we have $(P, \leq) = ([n], \cdot|\cdot)$. That is, $x \leq y$ if and only if $x$ divides $y$. We have seen examples of sums over this poset before; for instance, a theorem of Gauss states $n = \sum_{d|n} \varphi(d)$, where $\varphi$ is the Euler totient function. If we can compute the Möbius function for this poset, we could invert this sum and recover a formula for $\varphi(n)$.[11]

As luck would have it, we *can* compute the Möbius function, and shall do so below. To this end, we introduce a little notation. Given some $x \in \mathbb{N}$, let $p(x)$ denote the number of distinct prime divisors of $x$. For example, $p(17) = 1$, $p(30) = 3$ and $p(1024) = 1$. We also say a number is *squarefree* if it does not have a square divisor. In other words, every prime divisor divides it exactly once.

**Proposition 7.** *For the poset $([n], \cdot|\cdot)$, the Möbius function is*

$$\mu_{x,y} = \begin{cases} (-1)^{p(y/x)} & \text{if } x|y \text{ and } y/x \text{ is squarefree, and} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* If $x$ does not divide $y$, then $x \not\leq y$ in this poset, and so $\mu_{x,y} = 0$. Hence we may assume $x|y$ in what follows. We prove the proposition by induction on $y/x$.

For the base case, if $y/x = 1$, we have $y = x$, and by Theorem 1, $\mu_{x,x} = 1$. As $p(1) = 0$, this agrees with the formula above.

[11]Now of course you might argue that we already have a formula for $\varphi(n)$, so why jump through all these hoops? Well, it turns out that number theorists really like this poset[12] and sums of this form. If we compute the Möbius function just once, we will be able to invert all of those sums without any extra work.

[12]Unsurprisingly, their motto is "divide and conquer."

For the induction step, suppose we have a comparable pair $x, y$ with $y/x > 1$. Let $p(y/x) = a$, and suppose $p_1, p_2, \ldots, p_a$ are the $a$ distinct primes dividing $y/x$. By definition, $\mu_{x,y} = -\sum_{z:x|z,z|y,z\neq y} \mu_{x,z}$.

The induction hypothesis implies $\mu_{x,z} = 0$ unless $z/x$ is squarefree, which occurs when $z/x = \prod_{i\in I} p_i$ for some $I \subseteq [a]$. In this case, $z/x$ has $|I|$ distinct prime factors, and so the induction hypothesis gives $\mu_{x,z} = (-1)^{|I|}$.

If $y/x$ is not squarefree, then all such $z$ appear in the sum, and hence, splitting the sum by $k = |I|$,

$$\mu_{x,y} = -\sum_{I\subseteq[a]} (-1)^{|I|} = -\sum_{k=0}^{a} \sum_{I\subset[a],|I|=k} (-1)^{|I|} = -\sum_{k=0}^{a} \binom{a}{k} (-1)^k = -(1 + (-1))^a = 0,$$

as desired.

On the other hand, if $y/x$ is itself squarefree, then $y = x \prod_{i\in[a]} p_i$, so the term with $I = [a]$, or $k = a$ should be excluded from the sum. We then have

$$\mu_{x,y} = -\sum_{k=0}^{a-1} \binom{a}{k} (-1)^k = -(1 + (-1))^a + \binom{a}{a} (-1)^a = (-1)^a,$$

as given by the formula. This completes the induction step, and hence the proof. □

The formula in Proposition 7 may look somewhat familiar to you. If $y/x$ is squarefree with $a$ distinct prime factors, then we have $\mu_{x,y} = (-1)^a$. On the other hand, in the Boolean poset, if $S \subseteq T$ with $|T \setminus S| = a$, we have $\mu_{S,T} = (-1)^a$. This numerical agreement is no coincidence: the interval $[S, T] = \{R : S \subseteq R \subseteq T\}$ in the Boolean poset is isomorphic[13] to the interval $[x, y] = \{z : x|z \text{ and } z|y\}$ in the divisibility lattice. If we without loss of generality take $T\setminus S = [a]$, and have a prime factorisation $y/x = \prod_{i=1}^{a} p_i$, an isomorphism is given by $\Phi(S \cup I) = x \prod_{i\in I} p_i$ for all $I \subseteq [a]$. Since the Möbius function only depends on the poset, it follows that it takes the same values on isomorphic posets.

As before, knowledge of the Möbius function allows us to invert sums over the divisibility poset.

**Corollary 8.** *If we have a function $f : \mathbb{N} \to \mathbb{R}$, and $g : \mathbb{N} \to \mathbb{R}$ is given by $g(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n, n/d \text{ is squarefree}} (-1)^{p(n/d)} g(d)$.*

*Proof.* For fixed $n$, consider these functions over the divisibility poset $([n], \cdot|\cdot)$. Theorem 1 and Proposition 7 give

$$f(n) = \sum_{d|n} g(d)\mu_{d,n} = \sum_{d|n, n/d \text{ is squarefree}} (-1)^{p(n/d)} g(d).$$ □

For a final application, we will recover the formula for the Euler totient function.

[13]Two posets $(P_1, \leq_1)$ and $(P_2, \leq_2)$ are said to be *isomorphic* if there is a bijection $\Phi : P_1 \to P_2$ such that $x \leq_1 y \Leftrightarrow \Phi(x) \leq_2 \Phi(y)$.

**Corollary 9.** *If $n$ has prime factorisation $n = \prod_{i=1}^{a} p_i^{m_i}$, then the Euler totient function is given by*

$$\varphi(n) = n \prod_{i=1}^{a} \left(1 - \frac{1}{p_i}\right).$$

*Proof.* From the theorem of Gauss on the homework, we have the formula $n = \sum_{d|n} \varphi(d)$. We can obtain a formula for $\varphi(n)$ by applying Möbius inversion, where $f(n) = \varphi(n)$ is the totient function and $g(n) = n$ is the identity. We have

$$\varphi(n) = \sum_{d|n, n/d \text{ is squarefree}} (-1)^{p(n/d)} d.$$

If $d|n$ is such that $n/d$ is squarefree, then we must have $n/d = \prod_{i \in I} p_i$ for some subset $I \subset [a]$. Moreover, in this case $p(n/d) = |I|$. Hence

$$\varphi(n) = \sum_{I \subseteq [a]} (-1)^{|I|} \frac{n}{\prod_{i \in I} p_i} = n \sum_{I \subseteq [a]} \prod_{i \in I} \left(-\frac{1}{p_i}\right) = n \prod_{i \in I} \left(1 - \frac{1}{p_i}\right),$$

as required. □

## Conclusion

It is my hope that these examples show not just the utility of Möbius inversion, but also the power of mathematical abstraction. At first glance one might think that the determination of the totient function requires results from number theory, and the inclusion-exclusion principle set-theoretic arguments, and that the two have nothing to do with each other.

However, we have shown here that by generalising the problems to a higher level of abstraction, they both follow from the same thing: Möbius inversion. Moreover, when dealing with Möbius inversion over some arbitrary poset, we are very limited in what we can use — since we do not know what the poset in question is, we cannot use specific tools (which may be powerful but complicated) from that area. Instead, we were forced to find a simpler general argument, which in this case only relied upon inversion of matrices from linear algebra.

Given this general result, we were then able to derive the totient function and the inclusion-exclusion principle as almost immediate corollaries. Of course, once one descends from the abstract level to a more concrete example, one will need to use some properties of the specific setting in question to apply the general theorem, but this is usually much simpler.

In very general terms, while it is of course one of the goals of mathematics to classify which statements are true and which are not,[14] it is as important, if not more, to understand *why* theorems are true. Hence this practice of abstraction to simplify proofs and determine precisely what is needed and in what generality they hold is central to mathematics, and the Möbius inversion formula exemplifies this wonderfully.

[14] And, for those who are logically inclined, which statements are undecidable.