International Centre for Mathematical Sciences
Edinburgh, Scotland, UK
t: +44 (0)131 650 9831 e: info@icms.org.uk
w: www.icms.org.uk

~~Distribution Problems~~
Partition Problems

How?

n balls into k boxes

$\downarrow$ numbered/indistinguishable

$\downarrow$ numbered/indistinguishable

$\downarrow$ $\forall$ box non empty

$\forall$ box at most 1

arbitrary

· Distributing pennies to children

31 pennies - indistinguishable balls

3 children - numbered boxes

o o o o o | o o | o . . . o
A B C

Put down 33 objects (31 pennies + 2 dividers)

in a row $\binom{31+2}{2}$ (Choose positions of dividers)

In general: $\binom{n+k-1}{k-1}$

If all children should get at least one penny: $\binom{n-k+k-1}{k-1}$

k-multiset

Informal: repetition of elements allowed
$\{1,1,2,2,2,3\}$

Formal: non-decreasing sequence of k integers
$(1,1,2,2,2,3)$
$1\le 1\le 2\le 2\le 2\le 3$

# of k-multisets of an n-element set

$$\mathrm{Multi}\binom{[n]}{k}=\{(a_1,a_2,\dots,a_k):\ 1\le a_1\le a_2\le\dots\le a_k\le n\}\xleftrightarrow{F}\{(b_1,\dots,b_k):1\le b_1<b_2<\dots<b_k\le n+k-1\}$$

$$(a_1,\dots a_k)\xleftrightarrow{F}(a_1,a_2+1,a_3+2,\dots,a_i+i-1,\dots,a_k+k-1)$$

$$\updownarrow$$

$$\{a_1,a_2+1,\dots\quad a_k+k-1\}$$

$$\cap$$

$$\binom{[n+k-1]}{k}$$

$\boxed{F \text{ is bijection}}$

- $F(a_1,\dots,a_k)\in\binom{[n+k-1]}{k}$
  $1\le a_1<a_2+1<a_3+2<\dots<a_k+k-1\le n+k-1$
- F injective $(a_i\ne a_i'\Rightarrow a_i+i-1\ne a_i'+i-1)$
- F surjective $(1\le b_1<b_2<\dots\le b_k\le n+k-1$
  $\downarrow$
  $1\le b_1\le b_2-1\le b_3-2\le\dots\le b_k-k+1\le n)$

$$\left|\mathrm{Multi}\binom{[n]}{k}\right|=\left|\binom{[n+k-1]}{k}\right|=\binom{n+k-1}{k}$$

International Centre for Mathematical Sciences
Edinburgh, Scotland, UK
t: +44 (0)131 650 9831 e: info@icms.org.uk
w: www.icms.org.uk

n <u>different</u> balls to k <u>indisnguishable</u> boxes

s.t. NO Box is empty

<u>Def</u>: - partition of a set X is a collection of non-empty subsets of X s.t. each element belongs to exactly

⇕

$\{X_1, \dots, X_k\}$ s.t. $X_i \neq \emptyset \ \forall i$

$X_1 \cup \dots \cup X_k = X$

- $S(n,k)$ = # of k-partitions of $[n]$

i.e. $\left| \left\{ \{X_1, \dots, X_k\} : \begin{array}{l} X_1 \cup \dots \cup X_k = [n] \\ X_i \neq \emptyset \ \forall i = 1, \dots, k \end{array} \right\} \right|$

Stirling # of the <u>second</u> kind

<u>Remarks</u>:

$S(n,k) = 0 \qquad n < k$

$S(0,0) = 1$ (# of ways to distribute 0 objects into 0 boxes ONE : doing nothing)

<u>Example</u>: $S(n,1) = S(n,n) = 1$ ~~...~~ $[n]$

$[n] = \{1\} \cup \{2\} \cup \dots \cup \{n\}$

Example: $S(n, n-1) = \binom{n}{2}$

~~among the~~

Example: Calculate: $S(4,2) = 7$
Real Time exercise

Thm: $\forall\, k \leq n$

$$S(n,k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

Pf: Classify k-partitions of $[n]$ according to placement of n $\rightsquigarrow$ singleton $\{n\}$

$\rightsquigarrow$

$$\{\{X_1, \dots, X_k\} \in \mathcal{S}(n,k) : X_i \neq \{n\}\ \forall i = 1, \dots, k\}$$

$$\downarrow$$

$$\{X_1 \setminus \{n\}, \dots, X_k \setminus \{n\}\} \in \mathcal{S}(n-1, k)$$

International Centre for Mathematical Sciences
Edinburgh, Scotland, UK
t: +44 (0)131 650 9831 e: info@icms.org.uk
w: www.icms.org.uk

~~Ex~~ What if ~~balls~~ boxes ARE distinguishable?

~~Coroll~~

Corollary: $\underbrace{\#\left\{ f:[n] \longrightarrow [k] : \begin{array}{c} f^{-1}(i) \neq \emptyset \\ \forall i=1,\dots,k \end{array} \right\}}_{\text{surjective fns.}} = k!\, S(n,k)$

Pf: • First create partition of $[n]$ into k nonempty parts

• ~~Order~~ Then Assign parts to elements of $[k]$

Corollary: $\forall x \in \mathbb{C} \quad \forall n \in \mathbb{N}$

$$\underline{x^n} = \sum_{k=0}^{n} S(n,k)\, x^{\underline{k}}$$

Pf: Both sides are polynomials of degree n

They agree for all $x \in \mathbb{N}$ $\Longrightarrow$ Also agree for $x \in \mathbb{C}$

Left handside = # fcts from ~~[k]~~ $[n]$ to $[x]$

Classify fns according to size of image $(|f([n])|)$

if $|f([n])| = k \Longrightarrow \exists\ S(n,k)$ k-partitions of $[n]$

$\exists\ x(x-1)\dots(x-k+1)$ ways to assign images

International Centre for Mathematical Sciences
Edinburgh, Scotland, UK
t: +44 (0)131 650 9831 e: info@icms.org.uk
w: www.icms.org.uk

$1, x, x^2, \ldots, x^n, \ldots \in \mathbb{C}[x]$ is a basis of $\mathbb{C}[x]$

$1, x, x^{\underline{2}}, x^{\underline{3}}, \ldots, x^{\underline{n}}, \ldots \in \mathbb{C}[x]$ is also a basis

polynoms with coefficient from $\mathbb{C}$

$S_{n,k}$ provides the coefficients to transfer from the second to the first

$$
\begin{array}{c c}
 & k \\
n & \left(\begin{array}{ccccc}
1 & & & & \\
1 & 1 & & & \\
1 & 3 & 1 & & \\
1 & 7 & 6 & 1 & \\
1 & & & & \\
\vdots & & S(n,k) & & \\
\vdots & & & & \ddots
\end{array}\right)
\end{array}
\begin{pmatrix} 1 \\ x \\ x^{\underline{2}} \\ x^{\underline{3}} \\ \vdots \\ x^{\underline{n}} \\ \vdots \end{pmatrix}
=
\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \\ x^n \\ \vdots \end{pmatrix}
$$

What is the inverse of the Stirling matrix (of the second kind)

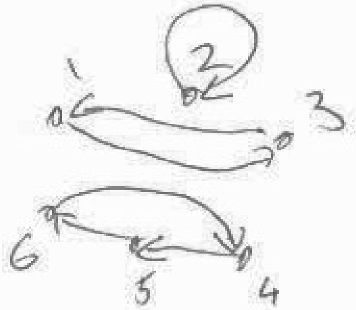
$\rightsquigarrow$ Stirling numbers of the first kind

Cycles in permutation

Recall n-permutation $\pi : [n] \to [n]$ bijective

Example
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 5 & 6 & 4 \end{pmatrix}$$
or write as word: 321564

DRAW



Group structure $(S_n, \circ)$ $\quad S_n = \{\pi : [n] \to [n] : \pi \text{ bijective}\}$

$\circ$ = composition

Example S_n is non-commutative alredy for $n=3$

Say 312 and 213

Lemma: $\forall \pi : [n] \to [n]$ perm $\forall x \in [n]$ $\exists i \in [n]$ s.t. $\pi^i(x) = x$

Pf: $\pi(x), \pi^2(x), \dots, \pi^n(x)$ if x is among them ✓

if x is not among them

$\Rightarrow \pi(x), \dots, \pi^n(x) \in [n] \setminus \{x\}$, n objects in $n-1$ boxes

$\Rightarrow \exists\, y \in [n] \setminus \{x\}$ and $j_1, j_2 \in [n]$ s.t. $\pi^{j_1}(x) = y = \pi^{j_2}(x)$

$\Rightarrow \pi^{j_1 - j_2}(x) = x$ $\quad j_1 - j_2 \in [n]$

~~Example: cycles of 321564~~

Def: ~~[scribbled out]~~ $\pi \in S_n$, $x \in [n]$

$$i = i(\pi, x) := \min \{ j \in [n] : \pi^j(x) = x \}$$

Then $x, \pi(x), \pi^2(x), \ldots, \pi^{i-1}(x)$ form an i-cycle in π

Corollary: All permutations can be decomposed into the disjoint union of cycles.

Example: Cycles of 321564 $(13)(2)(456)$

- (2) is a 1-cycle 2 is a fixed point
- (13) is a 2-cycle (transposition)

Prop
# of cyclic permutations $= (n-1)!$

has exactly one cycle

$$\boxed{(456) = (564) = (645)}$$

Pf: $(1 \_ \_ \_ \_ \ldots \_)$

1

What is the image of 1? $(n-1)$ answers

What is the image of the image of 1? $(n-2)$ answers

Def: Stirling number of the first kind

$s_{n,k}$ = # of n-permutations with exactly k-cycles

$$s_{n,0} := \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n = 1 \end{cases}$$

Examples: $s_{n,1} = (n-1)!$

$s_{n,n-1} = \binom{n}{2}$

$s_{n,n} = 1$

$s_{n,2} = (n-1)! \, H_n$

Real time exercise

1

0 1

0 1 1

0 2 3 1

0 6 11 6 1

0 24 50 35 10 1

Prop: $\sum_{k=0}^{n} s_{n,k} = n!$ $\forall n \geq k \geq 0$

Pf: Classify n-permutations according to # of cycles in cycle decomposition

The twelvefold ways of counting

How many ways are there to put

n numbered/indistinguishable balls

into r numbered/indistinguishable boxes

such that ~~each~~ $\forall$ box contains at most one ball (placement is injective)

— " — at least — " — (placement is surjective)

or the placement is arbitrary

	injective	surjective	arbitrary
balls numbered boxes — " —	$r^{\underline{n}}$	$r!\, S_{n,r}$	r^n
balls numbered boxes indistinguish	0 if $n>r$ 1 if $n\le r$	$S_{n,r}$	$\sum_{q=[illegible]}^{r} S_{n,q}$
balls indistinguishable boxes numbered	$\binom{r}{n}$	$\binom{n-1}{r-1}$	$\binom{n+r-1}{n}$
balls indistinguishable boxes — " —	0 if $n>r$ 1 if $n\le r$	$p(n,r)$	$\sum_{k=0}^{r} p(n,k)$

$$p(n,r) = \#\left\{(\lambda_1,\dots,\lambda_r) : \begin{array}{l}\lambda_1\ge\lambda_2\ge\dots\ge\lambda_r\ge 1\\ \lambda_1+\dots+\lambda_r = n\end{array}\right\}$$

"number partitions"

partition of n into r parts

$p(n) = \sum_{r=1}^{\infty} p(n,r) = \sum_{r=1}^{n} p(n,n)$ = # partitions of n into any number of parts