

Fibonacci sequence

- Leonardo of Pisa, 13th century, rabbits
- comes up often in nature and math/cs

$$F_0 = 0, F_1 = 1$$

$$\forall n \geq 2 \quad F_n = F_{n-1} + F_{n-2}$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Formula ???

Remark: Would be ~~easy~~ if $F_n = F_{n-1} + F_{n-1} \rightsquigarrow 2^n$
or $F_n = F_{n-2} + F_{n-2} \rightsquigarrow \sqrt{2}^n$

Idea

Encode sequence as ONE FUNCTION
(infinitely many numbers)

$$f(x) = \sum_{i=0}^{\infty} F_i x^i = F_0 + F_1 x + F_2 x^2 + \dots$$

How does this help?

Use recurrence \rightsquigarrow equation for the function

$$x f(x) = \sum_{i=0}^{\infty} F_i x^{i+1} =$$

$$F_0 x + F_1 x^2 + F_2 x^3 + F_3 x^4 + \dots$$

$$+ x^2 f(x) = \sum_{i=0}^{\infty} F_i x^{i+2} =$$

$$F_0 x^2 + F_1 x^3 + F_2 x^4 + \dots$$

$$x f(x) + x^2 f(x) = F_0 x + \sum_{i=2}^{\infty} (F_{i-2} + F_{i-1}) x^i = F_0 x + (F_0 + F_1) x^2 + (F_1 + F_2) x^3 + (F_2 + F_3) x^4 + \dots$$

Compare to

$$f(x) = \sum_{i=0}^{\infty} F_i x^i$$

$$= F_0 + F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \dots$$

$$x f(x) + x^2 f(x) - f(x) = F_0 x^0 - F_0 - F_1 x^1$$

$$f(x) = \frac{x}{1-x-x^2}$$

- Find series expansion (Taylor series)

Partial fractions: $x^2 - x - 1 = 0$
 $x_{1,2} = \frac{+1 \pm \sqrt{5}}{2}$ $\alpha_1 = \frac{-1 + \sqrt{5}}{2}$ $\alpha_2 = \frac{-1 - \sqrt{5}}{2}$

$$\frac{x}{1-x-x^2} = \frac{A}{1-\alpha_1 x} + \frac{B}{1-\alpha_2 x}$$

$$= \frac{x(-\alpha_2 A - \alpha_1 B) + A + B}{(1-\alpha_1 x)(1-\alpha_2 x)}$$

$$(1-\alpha_1 x)(1-\alpha_2 x) = 1 - (\alpha_1 + \alpha_2)x + \alpha_1 \alpha_2 x^2$$

$$= 1 - x - x^2$$

$$\Rightarrow -\alpha_2 A - \alpha_1 B = 1$$

$$A + B = 0$$

$$\Rightarrow -\alpha_2 A + \alpha_1 A = 1 \Rightarrow A = \frac{1}{\alpha_1 - \alpha_2} = -\frac{1}{\sqrt{5}}$$

$$B = \frac{1}{\sqrt{5}}$$

Express as power series (\rightarrow here: geometric series)

$$\frac{1}{1-\alpha_1 x} = \sum_{i=0}^{\infty} (\alpha_1 x)^i$$

$$\frac{1}{1-\alpha_2 x} = \sum_{i=0}^{\infty} (\alpha_2 x)^i$$

$$\sum_{i=0}^{\infty} F_i x^i = f(x) = \frac{x}{1-x-x^2} = A \sum_{i=0}^{\infty} (\alpha_1 x)^i + B \sum_{i=0}^{\infty} (\alpha_2 x)^i = \sum_{i=0}^{\infty} (A \alpha_1^i + B \alpha_2^i) x^i$$

$$\Rightarrow \forall i \in \mathbb{N} \quad F_i = \frac{1}{\sqrt{5}} \left(\frac{\sqrt{5}+1}{2} \right)^i - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^i$$

$$F_i = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^i - \left(\frac{1-\sqrt{5}}{2} \right)^i \right)$$

• Is this an integer ???

YES (we have proved it that it is the Fibonacci number)

• How big?

$$\frac{1}{\sqrt{5}} \left| \left(\frac{1-\sqrt{5}}{2} \right)^i \right| < \frac{1}{2} \Rightarrow$$

$$\Rightarrow F_i = \text{integer closest to } \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^i$$

Generating functions

Idea

Encode a sequence as a power series
(many numbers \leftrightarrow one function)

$$(a_0, a_1, a_2, \dots) \leftrightarrow a_0 + a_1x + a_2x^2 + \dots = \sum_{i=0}^{\infty} a_i x^i$$

Example:

$$(1, 1, 1, \dots, 1, \dots) \leftrightarrow \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

convergent for $|x| < 1$

Proposition

Let (a_0, a_1, a_2, \dots) be a sequence of reals.

If $|a_n| < k^n \quad \forall n \in \mathbb{N} \Rightarrow \forall x \in (-\frac{1}{k}, \frac{1}{k})$ $a(x) = \sum_{i=0}^{\infty} a_i x^i$
is absolute convergent.

$a(x)$ has derivatives of all order at 0

and for all $n \geq 0$ $a_n = \frac{a^{(n)}(0)}{n!}$

In particular, the values of $a(x)$ in an arbitrary small neighborhood of 0 determine ~~the~~ uniquely the sequence (a_0, a_1, a_2, \dots)

Remark: ~~The~~ Theory of formal power series

- ~~ANYWAYS~~: usually result (obtained by illegal methods) can be easily verified by induction

Def: (a_0, a_1, a_2, \dots) sequence of real numbers.
 The generating function of this sequence
 is the power series $a(x) = \sum_{i=0}^{\infty} a_i x^i$

Important examples:

$$(0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{i}, \dots) \leftrightarrow x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = -\ln(1-x)$$

(convergent for $\forall x \in (-1, 1)$)

$$(1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \dots, \frac{1}{i!}, \dots) \leftrightarrow \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$$

(convergent for $\forall x \in \mathbb{R}$)

$$\boxed{r \in \mathbb{R} !!!}$$

$$\left(\binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \binom{r}{3}, \dots, \binom{r}{i}, \dots \right) \leftrightarrow \sum_{i=0}^{\infty} \binom{r}{i} x^i = (1+x)^r$$

(convergent for $\forall x \in (-1, 1)$)

Special Case

$$\frac{1}{(1-x)^h} = (1-x)^{-h} = \sum_{i=0}^{\infty} \binom{-h}{i} (-1)^i x^i = \sum_{i=0}^{\infty} \binom{h-1+i}{h-1} x^i$$

Operations with sequences and their generating fns.
 Let (a_0, a_1, \dots) and (b_0, b_1, \dots) be sequences, $x \in \mathbb{C}$

(A) Addition

$$(a_0 + b_0, a_1 + b_1, \dots) \leftrightarrow a(x) + b(x) \quad \mathbb{N}$$

(B) Multiplication with a constant

$$(x a_0, x a_1, \dots) \leftrightarrow x a(x)$$

(C) Shifting to the right

$$(0, 0, \dots, 0, a_0, a_1, \dots) \leftrightarrow x^n a(x)$$

(D) Shifting to the left

$$(a_n, a_{n+1}, a_{n+2}, \dots) \leftrightarrow \frac{a(x) - a_0 - a_1 x - \dots - a_{n-1} x^{n-1}}{x^n}$$

(E) Substitute xx for x

$$(a_0, x a_1, x^2 a_2, x^3 a_3, \dots) \leftrightarrow a(xx)$$

Example: $(1, 2, 4, 8, \dots, 2^n, \dots) \leftrightarrow \frac{1}{1-2x}$

Application: Solving linear homogeneous recurrences with constant coefficients

There is partial fraction decomposition

$$a(x) = \frac{A_1}{1-\lambda_1 x} + \dots + \frac{A_2}{1-\lambda_2 x} = A_1 \sum (\lambda_1 x)^i + \dots + A_2 \sum (\lambda_2 x)^i$$

$$= \sum_{i=0}^{\infty} (A_1 \lambda_1^i + \dots + A_2 \lambda_2^i) x^i$$

provided the characteristic polynomial of the

$$\text{recurrence } \varphi(z) = z^2 - \alpha_{2-1} z^{2-1} - \dots - \alpha_0 z - \alpha_0$$

$$= (z - \lambda_1) \dots (z - \lambda_k)$$

has k distinct roots $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{C}$.

$$\left[\text{Then } (1 - \lambda_1 x) \dots (1 - \lambda_k x) = 1 - \alpha_{2-1} x - \dots - \alpha_0 x^2 \right.$$

$$\left. \begin{matrix} \iff \\ z = \frac{1}{x} \end{matrix} (z - \lambda_1) \dots (z - \lambda_k) = z^2 - \alpha_{2-1} z^{2-1} - \dots - \alpha_0 \right]$$

(Assume $\lambda_i \neq 0$, as then $\alpha_0 = 0 \Rightarrow$ recurrence is NOT degree 2)

Initial conditions $(a_0, a_1, \dots, a_{2-1})$
and coefficients $(\alpha_0, \alpha_1, \dots, \alpha_{2-1})$

determine $A_1, \dots, A_k \in \mathbb{C}$

and formula $\underline{a_n = A_1 \lambda_1^n + \dots + A_k \lambda_k^n} \quad \underline{\forall n \geq 0}$

Theorem: let $k \in \mathbb{N}$

• $x_0, \dots, x_{k-1} \in \mathbb{R}$

• sequence with recurrence

$$(*) a_n = x_{k-1} a_{n-1} + \dots + x_0 a_{n-k} \quad \boxed{\forall n \geq k}$$

• characteristic polynomial ϕ

$$p(x) = x^k - x_{k-1} x^{k-1} - \dots - x_0$$

$$= (x - \lambda_1) \dots (x - \lambda_k) \quad \text{with roots } \lambda_1, \dots, \lambda_k \in \mathbb{C}$$

(i) (Simple Roots)

Suppose

$$\lambda_i \neq \lambda_j \quad \forall i \neq j$$

Then for ANY sequence (a_0, a_1, a_2, \dots) satisfying $(*)$

$$\exists C_1, \dots, C_k \in \mathbb{C} \text{ s.t. } \boxed{\forall n \geq 0} C_1 \lambda_1^n + \dots + C_k \lambda_k^n = a_n$$

(ii) (General Case) HW

Suppose $p(x) = (x - \lambda_1)^{k_1} \dots (x - \lambda_q)^{k_q}$ with $k_1 + \dots + k_q = k$
and $\lambda_i \neq \lambda_j$ for $i \neq j$

Then for ANY sequence (a_0, a_1, \dots) satisfying $(*)$

$$\exists C_{ij} \quad \left(\begin{array}{l} \forall i=1, \dots, q \\ j=0, \dots, k_i-1 \end{array} \right) \text{ s.t. } \boxed{\forall n \geq 0} a_n = \sum_{i=1}^q \sum_{j=0}^{k_i-1} C_{ij} n^j \lambda_i^n$$

Def (i) $S = \left\{ (a_0, a_1, \dots) : a_n = \sum_{i=1}^k \alpha_{k-i} a_{n-i} \quad \forall n \geq k \right\}$

Observation: a_0, a_1, \dots, a_{k-1} determine uniquely the whole sequence (a_0, a_1, \dots)

• S is a vector space over \mathbb{C}

$(a_0, a_1, \dots), (b_0, b_1, \dots) \in S, \mu_a, \mu_b \in \mathbb{C}$

$\Rightarrow (\underbrace{\mu_a a_0 + \mu_b b_0}_{c_0}, \underbrace{\mu_a a_1 + \mu_b b_1}_{c_1}, \dots, \underbrace{\mu_a a_i + \mu_b b_i}_{c_i}, \dots) \in S$

Since $c_n = \mu_a a_n + \mu_b b_n = \mu_a \sum_{i=1}^k \alpha_{k-i} a_{n-i} + \mu_b \sum_{i=1}^k \alpha_{k-i} b_{n-i}$
 $= \sum_{i=1}^k \alpha_{k-i} (\underbrace{\mu_a a_{n-i} + \mu_b b_{n-i}}_{c_{n-i}})$

• $\dim S = k$

Canonical basis $e^{(0)} = (1, 0, \dots, 0, \overbrace{\alpha_0, \alpha_2 + \alpha_0, \dots}^{\text{determined}})$
 $e^{(1)} = (0, 1, 0, \dots, 0, \alpha_1, \dots)$
 \vdots
 $e^{(k-1)} = (0, 0, \dots, 1, \alpha_{k-1}, \dots)$

• linear independence ✓ (already the first k coordinates)

• generates $\forall (a_0, a_1, \dots) \in S$

$a_0 \cdot e^{(0)} + a_1 \cdot e^{(1)} + \dots + a_{k-1} \cdot e^{(k-1)}$

• Another basis!

$$(1, \lambda_i, \lambda_i^2, \dots, \lambda_i^{n-1}, \dots) \in S \quad \forall i=1, \dots, k$$

$$\lambda_i^n = \alpha_{n-1} \lambda_i^{n-1} + \dots + \alpha_0 \lambda_i^{n-2} \quad /: \lambda_i^{n-2}$$

$$\text{Since } \lambda_i^2 = \alpha_{n-1} \lambda_i^{n-1} + \dots + \alpha_0$$

$$\text{Since } p(\lambda_i) = 0$$

$$\Delta = \left\{ \begin{array}{l} (1, \lambda_1, \lambda_1^2, \dots, \lambda_1^{n-1}, \dots) \\ \vdots \\ (1, \lambda_k, \lambda_k^2, \dots, \lambda_k^{n-1}, \dots) \end{array} \right\} \text{ are lin. independent}$$

$$\text{Since } \det \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_k & \lambda_k^2 & \dots & \lambda_k^{k-1} \end{vmatrix} = \prod_{\substack{k \geq i > j \geq 1 \\ i > j}} (\lambda_i - \lambda_j) \neq 0$$

(Vandermonde determinant)

$|\Delta| = k \Rightarrow$ basis $\Rightarrow \forall (\alpha_0, \alpha_1, \alpha_2, \dots) \in S$
 can be expressed as linear combination
 $\Rightarrow \exists C_1, \dots, C_k \quad (\alpha_0, \alpha_1, \alpha_2, \dots) = \sum_{i=1}^k C_i (1, \lambda_i, \lambda_i^2, \dots)$

Pf of Vandermonde determinant

Induction on k

$$\det ||1|| = 1 = \prod_{i=1}^1 1 \quad \checkmark$$

$$\det \begin{vmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{vmatrix} = \lambda_2 - \lambda_1 \quad \checkmark$$

$k > 2$

$$\text{Let } Q(x) := \det \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{k-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_{k-1} & \lambda_{k-1}^2 & \dots & \lambda_{k-1}^{k-1} \\ 1 & x & x^2 & \dots & x^{k-1} \end{vmatrix}$$

$Q(x)$ is a polynomial with degree $\leq k-1$

• In fact $\deg Q(x) = k-1$, since coefficient of x^{k-1}

is $(-1)^{k(k-1)} \det \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{k-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_{k-1} & \lambda_{k-1}^2 & \dots & \lambda_{k-1}^{k-1} \end{vmatrix} = \prod_{k-1 \geq i > j \geq 1} (\lambda_i - \lambda_j)$

By induction \odot

• $\lambda_1, \dots, \lambda_{k-1}$ are all roots of $Q(x)$: we found all the roots!

$$\Rightarrow Q(x) = (x - \lambda_1) \dots (x - \lambda_{k-1}) \cdot \prod_{k-1 \geq i > j \geq 1} (\lambda_i - \lambda_j)$$

leading coefficient \odot

$$\Rightarrow Q'(\lambda_k) = \prod_{k \geq i > j \geq 1} (\lambda_i - \lambda_j)$$

(F) Substituting x^n for x

$$(a_0, \underbrace{0, \dots, 0}_{n-1}, a_1, \underbrace{0, \dots, 0}_{n-1}, a_2, 0, \dots) \Leftrightarrow a(x^n) = \sum_{i=0}^{\infty} a_i x^{ni}$$

Example: Generating fn of $a_n = 2^{\lfloor \frac{n}{2} \rfloor}$

That is $(1, 1, 2, 2, 4, 4, 8, 8, \dots)$

pp: $(1, 1, 1, \dots) \rightsquigarrow \frac{1}{1-x}$

Substitute x

$$(1, 2, 4, 8, \dots) \Leftrightarrow \frac{1}{1-2x}$$

Substitute x^2

$$(1, 0, 2, 0, 4, 0, 8, \dots) \Leftrightarrow \frac{1}{1-2x^2}$$

Shift to right by 1

$$(0, 1, 0, 2, 0, 4, 0, 8, \dots) \Leftrightarrow \frac{x}{1-2x^2}$$

$$\textcircled{1} + \textcircled{2} = (1, 1, 2, 2, 4, 4, 8, 8, \dots) \rightsquigarrow \frac{1+x}{1-2x^2}$$

(G) Differentiation / Integration

$$\frac{d}{dx} a(x) \longleftrightarrow (a_1, 2a_2, 3a_3, \dots, na_n, \dots)$$

$$\int a(x) dx \longleftrightarrow (0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2, \dots)$$

Example: generating fu of $a_n = (n+1)^2$

That is of $(1^2, 2^2, 3^2, \dots)$

$$(1, 1, 1, \dots) \longleftrightarrow \frac{1}{1-x}$$

Differentiate

$$(1, 2, 3, 4, \dots) \longleftrightarrow \frac{1}{(1-x)^2}$$

Differentiate

$$(1 \cdot 2, 2 \cdot 3, 3 \cdot 4, \dots) \longleftrightarrow \frac{2(1-x)}{(1-x)^4} = \frac{2}{(1-x)^3}$$

Subtract

$$(1^2, 2^2, 3^2, \dots) \longleftrightarrow \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}$$

H) Product of generating fns

$$a(x) \cdot b(x) \iff (c_0, c_1, c_2, \dots)$$

$$\text{where } c_0 = a_0 b_0$$

$$c_1 = a_0 b_1 + b_1 a_0$$

$$c_n = \sum_{k=0}^n a_k b_{n-k}$$

Proposition: ~~Let~~ $a_n := \#$ of ways to build a structure of Type I on ~~an set~~ $[n]$

$b_n := \#$ of ways to build a structure of Type II on $[n]$

$c_n := \#$ of ways to ~~se~~ first separate $[n]$ into two intervals $I = [i]$ and $[n] \setminus I$ and then build a structure of Type I on I and a structure of Type II on $[n] \setminus I$.

Then for the generating fns.

$$C(x) = A(x) \cdot B(x)$$

Pf: $\forall i = 0, 1, \dots, n$ $\exists a_i$ ways to build on $[i]$
and b_{n-i} " " " $[n] \setminus [i]$

$\Rightarrow a_i b_{n-i}$ to build on both

$$\Rightarrow c_n = \sum_{i=0}^n a_i b_{n-i}$$

Example: Design term in an Engineering Department
Term: n days

Decisions: How long should theoretical part be?
 k days ($1 \leq k \leq n-2$)

Rest is laboratory part
($n-k$ days)

- When should ~~the~~ be the
* project day for the theoretical part?
- * two project days for the lab part?

How many ways are there? $\boxed{f_n}$

Product Formula:

$a_n :=$ # ways to select project day from
an n day long theoretical part

$$a_n = n$$

$b_n :=$ # ways to select TWO project days from
an n day long lab part

$$b_n = \binom{n}{2}$$

$$a(x) = \sum_{n=1}^{\infty} n x^n = x \cdot \left(\sum_{n=0}^{\infty} x^n \right)' = x \cdot \left(\frac{1}{1-x} \right)' = \frac{x}{(1-x)^2}$$

$$b(x) = \sum_{n=2}^{\infty} \binom{n}{2} x^n = \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} x^n \right)'' = \frac{x^2}{2} \cdot \left(\frac{1}{1-x} \right)'' = \frac{x^2}{(1-x)^3}$$

$$f(x) = \sum_{n=0}^{\infty} f_n x^n = a(x) \cdot b(x) = \frac{x^3}{(1-x)^5} = x^3 \sum_{n=0}^{\infty} \binom{n+4}{4} x^n = \sum_{n=3}^{\infty} \binom{n+1}{4} x^n$$

$\rightarrow f_n = \binom{n+1}{4}$