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~~Distribution Problems~~

Partition Problems

How?

n balls into k boxes

↓ numbered/indistinguishable

↓ numbered/indistinguishable

↓ $\forall$ box non-empty

$\forall$ box at most 1

arbitrary

· Distributing pennies to children

31 pennies – indistinguishable balls

3 children – numbered boxes

$\circ\ \circ\ \circ\ \circ\ \circ \mid \circ\ \circ \mid \circ\ \_\_\_\_\ \circ$

A B C

Put down 33 objects (31 pennies + 2 dividers) in a row $\binom{31+2}{2}$ (Choose positions of dividers)

In general: $\binom{n+k-1}{k-1}$

If all children should get at least one penny: $\binom{n-k+k-1}{k-1}$

k- multiset — Informal: repetition of elements allowed
$\{1,1,2,2,2,3\}$

Formal: non-decreasing sequence of k integers
$(1,1,2,2,2,3)$
$1 \le 1 \le 2 \le 2 \le 2 \le 3$

# of k-multisets of an n-element set

$$\mathrm{Multi}\binom{[n]}{k} = \{(a_1, a_2, \dots, a_k) : 1 \le a_1 \le a_2 \le \dots \le a_k \le n\} \xleftrightarrow{F} \{(b_1, \dots, b_k) : 1 \le b_1 < b_2 < \dots < b_k \le n+k-1\}$$

$$(a_1, \dots, a_k) \xleftrightarrow{F} (a_1, a_2+1, a_3+2, \dots, a_i+i-1, \dots, a_k+k-1)$$
$$\updownarrow$$
$$\{a_1, a_2+1, \dots \quad a_k+k-1\} \in \binom{[n+k-1]}{k}$$

$\boxed{\text{F is bijection}}$

- $F(a_1, \dots, a_k) \in \binom{[n+k-1]}{k}$
  $1 \le a_1 < a_2+1 < a_3+2 < \dots < a_k+k-1 \le n+k-1$
- F injective $(a_i \ne a_i' \Rightarrow a_i+i-1 \ne a_i'+i-1)$
- F surjective $(1 \le b_1 < b_2 < \dots \le b_k \le n+k-1)$
  $\downarrow$
  $1 \le b_1 \le b_2-1 \le b_3-2 \le \dots \le b_k-k+1 \le n)$

$$\left|\mathrm{Multi}\binom{[n]}{k}\right| = \left|\binom{[n+k-1]}{k}\right| = \binom{n+k-1}{k}$$

n different balls to k indisnguishable boxes

s.t. NO Box is empty

Def: - partition of a set X is a collection of non-empty subsets of X s.t. each element belongs to exactly

$\Updownarrow$

$\{X_1, \dots, X_k\}$ s.t. $X_i \neq \emptyset \ \forall i$

$X_1 \cup \dots \cup X_k = X$

- $S(n,k)$ = # of k-partition of $[n]$

i.e. $\left| \left\{ \{X_1, \dots, X_k\} : \begin{array}{l} X_1 \cup \dots \cup X_k = [n] \\ X_i \neq \emptyset \ \forall i = 1, \dots, k \end{array} \right\} \right|$

Stirling # of the second kind

Remarks:

$S(n,k) = 0 \quad n < k$

$S(0,0) = 1$ (# of ways to distribute 0 objects into 0 boxes ONE: doing nothing)

Example. $S(n,1) = S(n,n) = 1 \quad$ ~~...~~ $[n]$

$[n] = \{1\} \cup \{2\} \cup \dots \cup \{n\}$

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Example: $S(n, n-1) = \binom{n}{2}$

Meaning:

$\boxed{\text{Example: Real Time exercise} \quad \text{Calculate: } S(4,2) = 7}$

Thm: $\forall\, k \leq n$

$$S(n,k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

Pf: Classify k-partitions of $[n]$ according to placement of n $\rightsquigarrow$ singleton $\{n\}$

$\rightsquigarrow$

$$\left\{ \{X_1, \dots, X_k\} \in \mathcal{S}(n,k) : X_i \neq \{n\} \ \forall\, i = 1, \dots, k \right\}$$

$$\downarrow$$

$$\{X_1 \setminus \{n\}, \dots, X_k \setminus \{n\}\} \in \mathcal{S}(n-1, k)$$

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~~#~~ What if boxes ARE distinguishable?

Corollary : $\# \underbrace{\{ f : [n] \to [k] : f^{-1}(i) \neq \emptyset \ \forall i = 1, \dots, k \}}_{\text{surjective fns.}} = k! \, S(n,k)$

Pf: • First create partition of [n] into k nonempty parts

• Then Assign parts to elements of [k]

Corollary : $\forall x \in \mathbb{C} \quad \forall n \in \mathbb{N}$

$$\underline{x^n} = \sum_{k=0}^{n} S(n,k) \, x^{\underline{k}}$$

Pf: Both sides are polynomials of degree n

They agree for all $x \in \mathbb{N}$ $\Longrightarrow$ Also agree for $x \in \mathbb{C}$

Left hand side = # fcts from [n] to [x]

Classify fns according to size of image $(|f([n])|)$

if $|f([n])| = k \Rightarrow \exists \ S(n,k)$ k-partitions of [n]

$\exists \ x(x-1) \dots (x-k+1)$ ways to assign were images

$1, x, x^2, \ldots, x^n, \ldots \in \mathbb{C}[x]$ is a basis of $\mathbb{C}[x]$

$1, x, x^{\underline{2}}, x^{\underline{3}}, \ldots, x^{\underline{n}}, \ldots \in \mathbb{C}[x]$ is also a basis

polynoms with coefficient from $\mathbb{C}$

$S_{n,k}$ provides the coefficients to transfer from the second to the first

$$\begin{array}{cc} & \begin{matrix} & & & & k \\ & & & & \downarrow \end{matrix} \\ n \to & \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & 0 & \\ 1 & 3 & 1 & & & \\ 1 & 7 & 6 & 1 & & \\ 1 & & & & \ddots & \\ \cdots & & S(n,k) & & & \ddots \\ \vdots & & & & & \end{pmatrix} \end{array} \begin{pmatrix} 1 \\ x \\ x^{\underline{2}} \\ x^{\underline{3}} \\ \vdots \\ x^{\underline{n}} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \\ x^n \\ \vdots \end{pmatrix}$$

What is the inverse of the Stirling matrix (of the second kind)

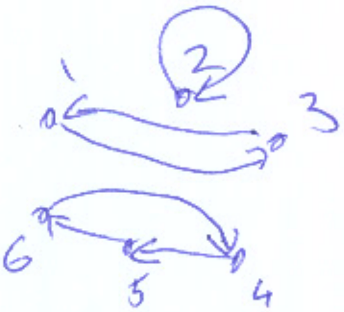
$\leadsto$ Stirling numbers of the first kind

Cycles in permutation

Recall n-permutation $\pi : [n] \to [n]$ bijective

Example $\overset{\pi}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 5 & 6 & 4 \end{pmatrix}}$ or write as word: 321564

DRAW



Group structure $(S_n, \circ)$ $\quad S_n = \{\pi : [n] \to [n] : \pi \text{ bijective}\}$

$\circ$ = composition

Example S_n is non-commutative alredy for $n=3$

Say 312 and 213

Lemma: $\forall \pi : [n] \to [n]$ perm $\forall x \in [n]$ $\quad \exists i \in [n]$ s.t. $\pi^i(x) = x$

Pf: $\pi(x), \pi^2(x), \dots, \pi^n(x)$ if x is among them ✓

if x is not among them

$\Rightarrow \pi(x), \dots, \pi^n(x) \in [n] \setminus \{x\}$, n objects in $n-1$ boxes

$\Rightarrow \exists\, y \in [n] \setminus \{x\}$ and $j_1, j_2 \in [n]$

s.t. $\pi^{j_1}(x) = y = \pi^{j_2}(x)$

$\Rightarrow \pi^{j_1 - j_2}(x) = x$ $\quad$ $j_1 - j_2 \in [n]$

~~Example: cycles of 321564~~

Def: $\pi \in S_n$, $x \in [n]$

$$i = i(\pi, x) := \min\{ j \in [n] : \pi^j(x) = x \}$$

Then $x, \pi(x), \pi^2(x), \dots, \pi^{i-1}(x)$ form an i-cycle in π

Corollary: All permutations can be decomposed into the disjoint union of cycles.

Example: Cycles of 321564 $(13)(2)(456)$

- (2) is a 1-cycle 2 is a fixed point
- (13) is a 2-cycle (transposition)

Prop # of cyclic permutations $= (n-1)!$

has exactly one cycle $\boxed{(456) = (564) = (645)}$

Pf: $(1 \sqcup \sqcup \sqcup \quad \sqcup)$

1

What is the image of 1? $(n-1)$ answers

What is the image of the image of 1? $(n-2)$ answers

Def: Stirling number of the first kind

$s_{n,k}$ = # of n-permutations with exactly k-cycles

$$s_{n,0} := \begin{cases} 0 & \text{if } n>0 \\ 1 & \text{if } n=1 \end{cases}$$

Examples: $s_{n,1} = (n-1)!$

$$s_{n,n-1} = \binom{n}{2}$$

$$s_{n,n} = 1$$

$$s_{n,2} = (n-1)! \, H_n$$

Real time exercise

1
0 1
0 1 1
0 2 3 1
0 6 11 6 1
0 24 50 35 10 1

Prop: $\displaystyle\sum_{k=0}^{n} s_{n,k} = n! \qquad \forall\, n \geq 0$

Pf: Classify n-permutations according to # of cycles in cycle decomposition

Prop: $s_{n,k} = s_{n-1,k-1} + (n-1)\, s_{n-1,k}$

Pf: Classify according to cycle of n:

Case 1, (n) is a cycle of length 1

$S(n-1,k-1)$ $\quad \pi \leadsto \pi$ with (n) deleted is an $(n-1)$-permutation with $k-1$ cycles

Case 2: Map F

$$\pi = (\dots n x \dots)(\dots)\dots \xrightarrow[\text{Delete } n]{} (\dots x \dots)(\dots)\dots$$

$\downarrow$

$(n-1)$ perm with k cycles

$(n-1)\, s(n-1,k)$

$\forall (n-1)$-permutation with k cycles occurs $(n-1)$ times as the image under F, because n can be placed in front of ANY of the $n-1$ symbols, each time creating a different n-permutations with k cycles. $\square$

Prop: $x^{\overline{n}} = \sum_{k=0}^{n} s_{n,k}\, x^k \quad \forall n \geq 0$

Pf: Induction on n $\quad n=1 \leadsto x = s_{1,0} \cdot x^0 + s_{1,1}\, x$ ✓

$\boxed{n>1}$ $x^{\overline{n}} = x^{\overline{n-1}} \cdot (x+n-1) = x \cdot x^{\overline{n-1}} + (n-1)\, x^{\overline{n-1}} = x \cdot \sum_{k=0}^{n-1} s_{n-1,k}\, x^k + (n-1) \sum_{k=0}^{n-1} s_{n-1,k}\, x^k$

$$= \sum_{k=0}^{n-1} s_{n-1,k}\, x^{k+1} + (n-1) \sum_{k=0}^{n-1} s_{n-1,k}\, x^k =$$

$$= \sum_{j=1}^{n} s_{n-1,j-1}\, x^j + \sum_{k=1}^{n} (n-1)\, s_{n-1,k}\, x^k =$$

$$= \sum_{j=1}^{n} \underbrace{\left(s_{n-1,j-1} + (n-1)\, s_{n-1,j}\right)}_{s_{n,j}} x^j = \sum_{j=0}^{n} s_{n,j}\, x^j$$

Prop: $x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s_{n,k}\, x^k$

Pf:

$$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}} = (-1)^n \sum_{k=0}^{n} s_{n,k} (-x)^k$$

$$= \sum_{k=0}^{n} s_{n,k} (-1)^{n+k} x^k$$

signed Stirling numbers of the first kind

Corollary: $\forall\, i,j$

$$\sum_{k=j}^{i} (-1)^{k-j} S_{i,k}\, s_{k,j} = \delta_{i,j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

(The inverse of the Stirling matrix of the second kind is the signed Stirling matrix of the first kind.)

The twelvefold ways of counting

How many ways are there to put

n numbered/indistinguishable balls

into r numbered/indistinguishable boxes

such that ~~each~~ $\forall$ box contains at most one ball (placement is injective)

-"- at least -"- (placement is surjective)

or the placement is arbitrary

	injective	surjective	arbitrary
balls numbered boxes -"-	$r^{\underline{n}}$	$r!\, S_{n,r}$	r^n
balls numbered boxes indistinguish	0 if $n>r$ 1 if $n\le r$	$S_{n,r}$	$\sum_{q=0}^{r} S_{n,q}$
balls indistinguishable boxes numbered	$\binom{r}{n}$	$\binom{n-1}{r-1}$	$\binom{n+r-1}{n}$
balls indistinguishable boxes -"-	0 if $n>r$ 1 if $n\le r$	$p(n,r)$	$\sum_{k=0}^{r} p(n,k)$

$$p(n,r) = \#\left\{(\lambda_1,\dots,\lambda_r) : \begin{array}{l}\lambda_1\ge\lambda_2\ge\dots\ge\lambda_r\ge 1\\ \lambda_1+\dots+\lambda_r=n\end{array}\right\}$$

partition of n into r parts

"number partitions"

$$p(n)=\sum_{r=1}^{\infty} p(n,r)=\sum_{r=1}^{n} p(n,r) = \#\text{partitions of } n \text{ into any number of parts}$$

~~[illegible]~~ Small values (Real Time Exercise)

$n \backslash k$	0	1	2	3	4			
0	1							
1	0	1						
2	0	1	1					
3	0	1	1	1				
4	0	1	2	1	1			
5	0	1	2	2	1	1		
6	0	1	3	3	2	1	1	
7	0	1	3	4	3	2	1	1

$p(0,0) := 1$

$p(n,0) = 0 \quad \forall n > 0$

- $p(n,1) = 1$
- $p(n,n) = 1$
- $p(n,n-1) = 1$
- $p(n,n-2) = 2$ $\quad 1\dots122$, $1\dots113$

Solve 6^{th} row $\quad p(6) = 11$

<u>Prop</u> $\quad p(n,k) = p(n-1,k-1) + p(n-k,k)$

<u>Pf</u>: Classify according to whether the smallest part is 1 or NOT

# of these is $p(n-1,k-1)$ $\qquad$ # of these is $p(n-k,k)$

Bijection

$$\lambda_1 \geq \dots \geq \lambda_k \geq 2 \longleftrightarrow \lambda_1 - 1 \geq \lambda_2 - 1 \geq \dots \geq \lambda_k - 1 \geq 1$$

$$\sum_{i=1}^{n} \lambda_i = n \qquad \sum_{i=1}^{k} (\lambda_i - 1) = n-k$$