Discrete Mathematics I So 18
Tibor Szabó
Tamás Mészáros
Anurag Bishnoi

Exercise Sheet 10

Due date: July 3rd at 4:15 PM

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Let $G = (V, E)$ be a connected graph on at least two vertices, and $\omega : E \to \mathbb{R}$ a weighting of its edges. For each $v \in V$, let ω_v be the minimum weight of an edge incident to the vertex v. Prove that for every $v \in V$, every minimum weight spanning tree of G contains an edge e such that $\omega(e) = \omega_v$.

Exercise 2 Show that a tree has at most one perfect matching.

Exercise 3 In the lecture we proved that every graph on $n \geq 3$ vertices in which the minimum degree is at least $n/2$ is Hamiltonian. Use the same ideas to prove the following generalization.

(1) Prove that a (simple) graph G on $n \geq 3$ vertices is Hamiltonian if for every pair of non-adjacent vertices u, v in G we have $\deg(u) + \deg(v) \geq n$.

(2) Conclude that the number of edges in a non-Hamiltonian graph on n vertices is at most $\binom{n-1}{2} + 1$.

Exercise 4 A TV remote accepts two batteries, but will only function if both of the batteries are charged. There are nine batteries, of which only three are charged, and the other six have no charge remaining. However, there is no way to distinguish between the charged and drained batteries other than putting them in the remote and seeing if it works. Determine the minimum number of pairs of batteries that must, in the worst case, be tested in order to get the remote working.

Exercise 5 Consider an 8×8 chess board in which each row and each column has exactly 4 rooks. Prove that we can choose 8 rooks out of these such that no two of them attack each other.

Exercise 6 Determine the maximum number of edges that an n-vertex graph can have without containing a $K_{1,4}$ as a subgraph.