Discrete Mathematics I — So 18

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Exercise Sheet 12

Due date: July 17th at 4:15 PM

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Given a graph G, let $P(G,k)$ denote the number of proper k-colourings of G.

(1) Prove that $P(G,k) = P(G-e,k) - P(G/e,k)$ for every edge e in G, where G/e is the graph obtained from G be identifying the two endpoints of e as one vertex.

(2) Prove that $P(G,k)$ is a polynomial in k of degree $|V(G)|$.

Exercise 2 Let G be a planar graph on n vertices that has girth g. Prove that G has at most $g(n-2)/(g-2)$ edges. Deduce that the Petersen graph is non-planar.

Exercise 3

(1) Show that any graph with $\chi(G) = k$, must have at least edges $\binom{k}{2}$ edges.

(2) Deduce that if a graph G is the (not necessarily disjoint) union of m copies of K_m, then $\chi(G) \leq m^{3/2}$.

Exercise 4 Let S be a set of n points in the plane such that any two distinct points in S are at distance at least 1 from each other. Prove that the number of pairs of points which are at distance *exactly* 1 from each other is at most $3n-6$.

Exercise 5 A planar graph G is outerplanar if there is an embedding of it in the plane such that all vertices are on the boundary of the outer face. Use Kuratowski's Theorem to show that a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

Exercise 6 Prove that a set of edges in a connected plane graph G forms a spanning tree of G is and only if the duals of the remaining edges form a spanning tree of G^*.