Discrete Mathematics I So 18
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Exercise Sheet 2

Due date: May 2nd at 10:30 AM

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Give combinatorial proofs of the following identities:

(a) For all $m, n \in \mathbb{N}$, and $k \geq 0$,

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}.$$

(b) For all $n \in \mathbb{N}$ and $0 \leq k \leq n$,

$$\binom{n}{k} = \sum_{i=0}^{k} \binom{n-i-1}{k-i}.$$

Exercise 2 There are 20 kids who want to get ice cream cones. There are 5 different flavours in which the cones are available. What is the total number of ways of distributing ice cream cones, one cone per kid, in the following cases:

(a) each kid gets one scoop of ice cream and all flavours are used;

(b) each kid can choose either one or two scoops, of any variation of flavours, for their ice cream cone. [1]

Exercise 3 Let F be a field and let $F[x_1, \ldots, x_n]$ denote the ring of n-variable polynomials over F. A polynomial $f \in F[x_1, \ldots, x_n]$ is called homogeneous if every term appearing in f has the same degree, i.e., the degree of f. For example, $x^2y + xy^2 + x^3$ is a homogeneous polynomial of degree 3 in $F[x, y, z]$.

Given a positive integer d, the linear combination of two homogeneous polynomials of degree d is also a homogeneous polynomial of degree d, and hence the set $V_{n,d}$ of homogeneous polynomials of degree d in $F[x_1, \ldots, x_n]$ form a vector space.[2] Determine the dimension of $V_{n,d}$.

[1] the order of flavours on a cone doesn't matter
[2] you are welcome to prove this

Exercise 4 Prove the following exact formulae for Stirling numbers of the second kind for all integers $n \geq 1$:

(a) $S(n, 2) = 2^{n-1} - 1$.

(b) $S(n, 3) = \frac{1}{6}(3^n - 3 \cdot 2^n + 3)$.

Exercise 5 Let $s_{n,k}$ denote the Stirling number of the first kind. Prove that for every $n \geq 1$, there is some $m(n)$ such that

$$s_{n,0} < s_{n,1} < \cdots < s_{n,m(n)-1} \leq s_{n,m(n)} > s_{n,m(n)+1} > \cdots > s_{n,n}.$$[3]

Moreover, either $m(n) = m(n-1)$ or $m(n) = m(n-1) + 1$.

[3] such sequences which have a single maximum/minimum are called unimodular