Discrete Mathematics I So 18
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Exercise Sheet 3

Due date: May 8th at 4:15 PM

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Prove that the average number of cycles in a permutation of length n is exactly H_n, where $H_n = \sum_{k=1}^{n} 1/k$.

Exercise 2 Let F_n denote the n-th Fibonacci number.

(a) Prove that the number of ways of tiling a $1 \times n$ rectangular grid using squares and dominoes is equal to F_{n+1}, where $n \geq 1$. [1]

(b) Give a combinatorial proof of the fact that

$$\sum_{k=1}^{n} F(k) = F(n+2) - 1.$$

(c) Give a combinatorial proof of the fact that:

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}.$$

Exercise 3 For $n \geq 1$, let t_n denote the number of elements in $\{0, 1, 2\}^n$ which have the property that they never contain a 2 followed immediately by a 0. For example, $t_2 = 8$ since all strings of length 2 except the string $(2, 0)$ are in the set that we are counting.

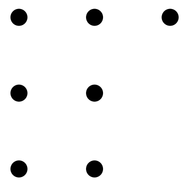
(a) Find a recurrence relation that the sequence (t_n) satisfies.

(b) Give an explicit formula for t_n.

[1] This formulation of the Fibonacci numbers appears in the work of the Indian linguist Virahanka (700 AD) and the poet Hemachandra (1150 AD), where they were interested in counting the number of patterns of length n that can be formed by using short syllables (squares) or long syllables (dominoes).

Exercise 4 Recall that the number of distributions of n indistinguishable balls into k indistinguishable boxes, with no restrictions, is equal to $\sum_{i=0}^{k} p(n,i)$ where $p(n,i)$ is the number of such distributions in which exactly i boxes are non-empty. Prove that this number is equal to the total number of integer partitions of a positive integer n into parts of size at most k, i.e., the number of ways in which n can be written as $n = \lambda_1 + \cdots + \lambda_r$ with $k \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 1$, for some positive integer r and integers $\lambda_1, \ldots, \lambda_r$.

Exercise 5 An integer partition of n into k parts as $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ can be visually represented by putting n dots into k rows where the i-th row has λ_i dots in it. For example, for $7 = 3 + 2 + 2$ we have the following picture:



This is known as the *Ferrers diagram* of the integer partition. The conjugate of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, denoted by λ^*, corresponds to the partition one gets by reflecting the Ferrers diagram along the line $y = -x$ (assuming that the upper left dot has coordinate $(0,0)$. A partition is called self-conjugate if $\lambda = \lambda^*$. For example, $6 = 3 + 2 + 1$ is a self conjugate partition.

Prove that the number of self-conjugate partitions of n (into any number of parts) is equal to the number of partitions of n in which each part has odd size and no two parts are equal.