Discrete Mathematics I So 18
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Exercise Sheet 4

Due date: May 15th at 4:15 PM

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Determine a closed form for the generating functions, $A(x)$, of the following sequences.

(i) $a_n = n^3$ for all $n \geq 0$.

(ii) $a_n = \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 2^n + 3^{n/2} & \text{if } n \text{ is divisible by 4} \\ 2^n - 3^{n/2} & \text{if } n \text{ is even, but not divisible by 4} \end{cases}$.

Exercise 2 Show that $(7 + \sqrt{50})^n$ has at least n zeroes following the decimal point, for every odd n.

[Hint (to be read backwards): .noitseuq ni rebmun siht ot esolc si hcihw fo noitulos eht noitaler ecnerrucer a pu tes ot yrT]

Exercise 3 In this exercise we will see how Catalan numbers pop up all over the place.[1] Prove that both of the following sequences are equal to the sequence $(c_n)_{n\geq 0}$ of Catalan numbers.

(a) $(t_n)_{n\geq 0}$, where t_n is the total number of triangulations of a convex $(n+2)$-gon, i.e., the number of ways a convex polygon with $n+2$ sides can be cut into triangles by connecting its vertices with straight non-crossing lines [2]. See Figure 1 for the case $n = 4$.

[1]Indeed, Richard Stanley's *Enumerative Combinatorics: Volume 2* famously has a set of exercises with no fewer than 66 different Catalan structures!

[2]we assume $t_0 = 1$ since there is no convex 2-gon, and hence we have nothing to do, and we can do nothing in exactly one way

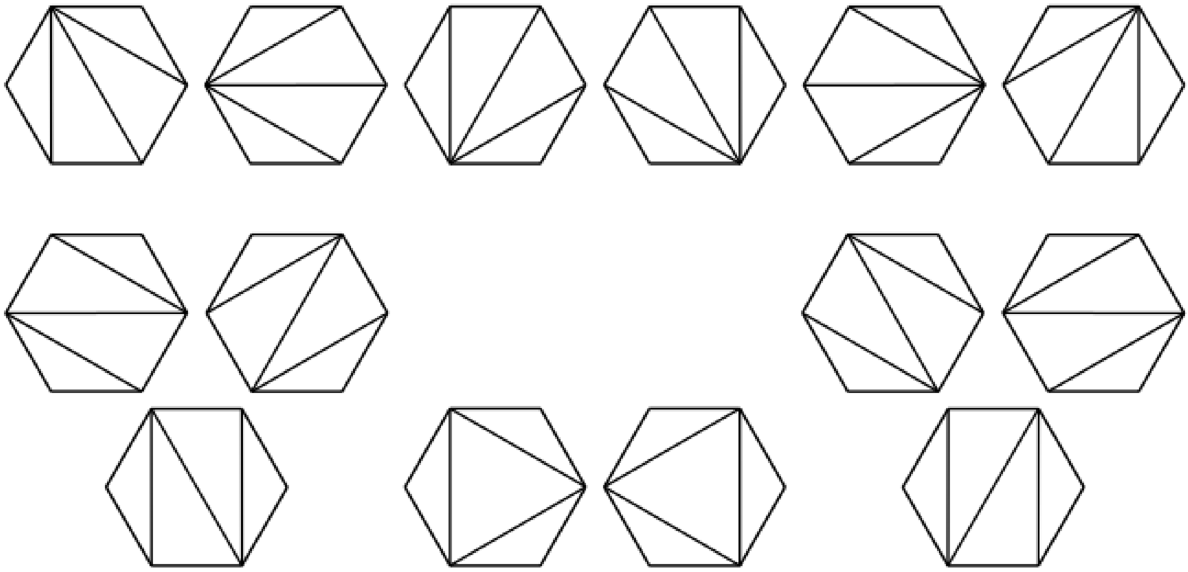

Figure 1: Triangulations of hexagons. (Thanks, Wikipedia!)

(b) $(h_n)_{n\geq 0}$, where h_n is the number of ways in which $2n$ people sitting around a table can shake hands with each other such that no two handshakes cross. For example, if Aida, Benny, Carl and Danaë are sitting around a table in that order, then we have the following two possible arrangements of handshakes: {{Aida, Benny}, {Carl, Danaë}} and {{Aida, Danaë}, {Benny, Carl}}.

(c) (Bonus) $(p_n)_{n\geq 0}$, where p_n is the total number of permutations π of the numbers $\{1, \ldots, n\}$ which has no three term increasing subsequence, i.e., there does not exist $1 \leq i < j < k \leq n$ such that $\pi(i) < \pi(j) < \pi(k)$. For example, we have the $p_4 = 14$ since the set of valid permutations is given by
$\{1432, 2143, 2413, 2431, 3142, 3214, 3241, 3412, 3421, 4132, 4213, 4231, 4312, 4321\}$.

Exercise 4 Given complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, consider the $n \times n$ matrix

$$M = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{n-1} \end{pmatrix},$$

where $M_{i,j} = \lambda_i^{j-1}$. Prove the Vandermonde formula:

$$\det(M) = \prod_{1\leq i<j\leq n} (\lambda_j - \lambda_i).$$

Exercise 5 In this exercise, you will give a linear algebraic proof of the solution to constant-coefficient linear homogeneous recurrence relations.

Suppose the sequence $(a_n)_{n\geq 0}$ satisfies the recurrence relation

$$a_n = c_{d-1}a_{n-1} + c_{d-2}a_{n-2} + \ldots + c_1 a_{n-d+1} + c_0 a_{n-d},$$

with $c_0 \neq 0$. Suppose further that the characteristic polynomial $P(x) = x^d - c_{d-1}x^{d-1} - \ldots - c_1 x - c_0$ has distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_d$.

(a) Show that the set of sequences satisfying the recurrence relation forms a vector space.[3]

(b) Show that the sequences $(\lambda_i^n)_{n\geq 0}$ are solutions, for every $1 \leq i \leq d$.

(c) Prove that the sequences in (b) form a basis for the space of solutions.

(d) Prove that, for any d initial values $a_n = \alpha_n$ for $0 \leq n \leq d-1$, there is a unique set of coefficients $\beta_1, \beta_2, \ldots, \beta_d$ such that $a_n = \sum_{i=1}^{d} \beta_i \lambda_i^n$ for all $n \geq 0$.

Exercise 6 In this exercise, you will determine the general solution to recurrence relations where repeated roots are allowed.

(a) Suppose the sequence $(b_n)_{n\geq 0}$ has the generating function $B(x) = (1 - \lambda x)^{-m}$. Show that the sequence is given by $b_n = \binom{m+n-1}{n}\lambda^n$.

(b) Let $(a_n)_{n\geq 0}$ be a sequence determined by a recurrence relation with characteristic polynomial $P(x) = \prod_{i=1}^{r}(x - \lambda_i)^{m_i}$; that is, the characteristic polynomial has distinct roots λ_i, each appearing with multiplicity m_i. Show that the solution must take the form[4]

$$a_n = \sum_{i=1}^{r} \left(\sum_{j=1}^{m_i} \beta_{i,j} \binom{n-1+j}{n} \right) \lambda_i^n,$$

where the coefficients $\beta_{i,j}$ can be determined from the initial conditions.

[3] This would be a subspace of $\mathbb{C}^{\mathbb{N}\cup\{0\}}$, so you only need to show it is closed under linear combinations.

[4] The solution is often presented in the more convenient (but equivalent) form $\sum_{i=1}^{r} \left(\sum_{j=0}^{m_i-1} \tilde{\beta}_{i,j} n^j \right) \lambda_i^n$.