Discrete Mathematics I
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Exercise Sheet 5

Due date: May 22 at 4:15 PM

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Find the generating function of the sequence $(a_0, a_1, \ldots)$ given by the recurrence relation $6a_n = 3a_{n-1} - 2a_{n-2} + a_{n-3}$ for every $n \geq 3$ and the initial values $a_0 = 14$, $a_1 = 16$, and $a_2 = 14$ and use it to derive a closed formula for a_n.

Exercise 2 Given sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, let $\hat{A}(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ and $\hat{B}(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ be their respective exponential generating functions. Let λ, λ_1 and λ_2 be some constants in $\mathbb{C}$. If $\hat{C}(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!}$, determine c_n in terms of $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ in the following cases.

(a) $\hat{C}(x) = \lambda_1 \hat{A}(x) + \lambda_2 \hat{B}(x)$ (b) $\hat{C}(x) = \hat{A}(\lambda x)$ (c) $\hat{C}(x) = x\hat{A}(x)$
(d) $\hat{C}(x) = \hat{A}(x^2)$ (e) $\hat{C}(x) = \int_0^x \hat{A}(t)\, dt$

Exercise 3 Let t_n be the number of ways to arrange n books on two bookshelves such that each shelf contains at least one book. Use generating functions to derive a closed formula for t_n. Verify your answer by giving a direct combinatorial proof.

Exercise 4 A *derangement* of $[n]$ is a permutation σ with no fixed points, that is, $\sigma(i) \neq i$ for all $i \in [n]$. Let d_n denote the number of derangements of $[n]$.

(a) Prove that $n! = \sum_{i=0}^{n} \binom{n}{i} d_{n-i}$.

(b) Use (a) to obtain the exponential generating function of the sequence $(d_n)_{n\geq 0}$.

(c) Prove that the probability that a uniformly random permutation of $[n]$ is a derangement is equal to

$$\sum_{i=0}^{n} (-1)^i \frac{1}{i!}.$$

[Hint (to be read backwards): ?ecneuqes lanigiro eht fo smus laitrap fo ecneuqes eht fo noitcnuf gnitareneg yranidro eht si tahw ,ecneuqes a fo noitcnuf gnitareneg yranidro eht neviG]

Exercise 5 Recall the following problem from the lecture. We have $2n$ distinct cards, for $n \geq 1$, and we split them into non-empty groups so that each of them contains an even number of cards. Then we order the cards within each subgroup and finally we order these subgroups into a line. Give a generating function free proof of the fact that the number of ways of doing so is $2^{n-1} \cdot (2n)!$.