Discrete Mathematics I — So 18

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Exercise Sheet 6

Due date: May 29 at 4:15 PM

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 For each pairs of functions $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ below, determine whether $f = o(g)$, $g = o(f)$, $f = O(g)$, $f = \Omega(g)$, $f = \Theta(g)$ or $f \sim g$ (as the argument n tends to infinity). The function log below denotes the logarithm with base 2.

(1) $f(n) = n^{1/\log n}$ and $g(n) = \log n$.

(2) $f(n) = 2^{n^2}$ and $g(n) = n!$.

(3) $f(n) = k\binom{n}{k}$ and $g(n) = k^2 2^{k \log n}$, where k is a constant.

Exercise 2 Four couples are planning to play a game of Coup[1]. In how many ways can they sit around a table so that no couple sits together?[2]

Exercise 3 For $n \in \mathbb{N}$, let p_n denote the number of permutations σ of $[n]$ which satisfy $\sigma(\sigma(i)) = i$ for all $i \in [n]$. Define $p_0 = 1$. Prove the recurrence $p_n = p_{n-1} + (n-1)p_{n-2}$, for all $n \geq 2$, and find the exponential generating function of the sequence.

Exercise 4 Let t_n, for $n \geq 1$, be the number of elements of $\{0, 1, 2\}^n$ in which there are an odd number of 0's and an odd number of 1's. Use generating functions to find a closed formula for this sequence.

[1] https://boardgamegeek.com/boardgame/131357/coup

[2] With all the lying and deception involved in the game, it's often best to avoid couples sitting too close to each other.

Exercise 5 Let $\pi(n) = |\{p \in [n] : p \text{ is prime}\}|$ be the prime number function, counting the number of primes in $[n]$. In this exercise you will determine the order of magnitude of $\pi(n)$.[3]

(a) Show that for every $m \in \mathbb{N}$ and every prime $p \in [m+1, 2m]$, $p | \binom{2m}{m}$.

(b) Deduce $\pi(n) = O\left(\frac{n}{\ln n}\right)$.

(c) Show that if p^k is a prime power such that $p^k | \binom{2m}{m}$, then $p^k \leq 2m$.

(d) Deduce $\pi(n) = \Omega\left(\frac{n}{\ln n}\right)$.

[3] You are asked to show $\pi(n) = \Theta\left(\frac{n}{\ln n}\right)$. However, more is known. The distribution of the prime numbers has long been central to number theory. Indeed, it was around 1800 that the legendary Legendre conjectured $\pi(n) \approx \frac{n}{\ln n - 1.08366}$. A similar conjecture was made by Gauss around the same time (when he was no older than 16).

In 1850, Chebyshev proved that $\frac{n}{\ln n}$ was the correct order of magnitude, and in 1896, Hadamard and de la Vallée Poussin independently extended the work of Riemann and proved the Prime Number Theorem, which gives the asymptotics of $\pi(n)$. As conjectured, $\pi(n) \sim \frac{n}{\ln n}$. These proofs all made use of complex analysis.

Since then, several other proofs have been found. Around 1950, Selberg and Erdős found elementary (i.e. not using analysis) proofs. (There was a rather bitter dispute between the two regarding who should get credit for the result.) The simplest proof currently known is due to Newman, although this also uses some complex analysis.