Discrete Mathematics I — So 18
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Exercise Sheet 7

Due date: June 5 at 4:15 PM

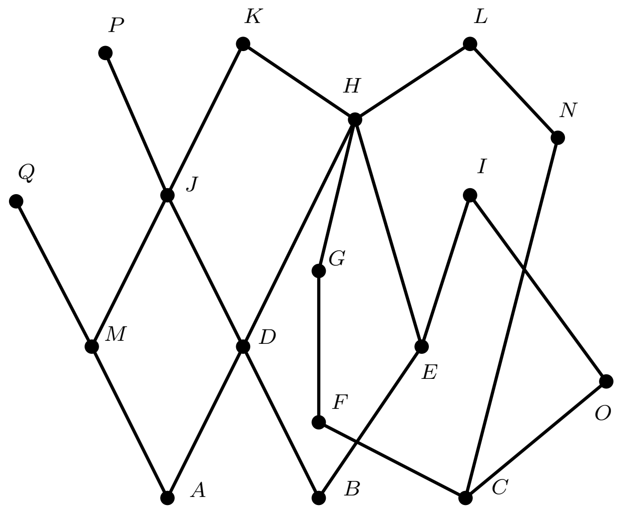
You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 What is the number of integers smaller than one million that contain two consecutive digits which are the same?

Exercise 2 Let $S(n, k)$ denote the Stirling number of the second kind. Give a combinatorial proof of the following identity:

$$k!S(n,k) = \sum_{i=0}^{k} (-1)^{(k-i)} i^n \binom{k}{i}.$$

Exercise 3 Find the width and the height of the poset given by the following Hasse diagram, and prove the correctness of your answer.



Exercise 4 Let $\mathcal{B}_n = (2^{[n]}, \subseteq)$ be the Boolean poset of all subsets of $[n]$.

(a) Find the smallest possible chain partition of $\mathcal{B}_4$.

(b) A symmetric chain in $\mathcal{B}_n$ is a chain $S_k, S_{k+1}, \ldots, S_{n-k}$ such that $|S_i| = i$ for all i. Prove that for all $n \geq 1$, $\mathcal{B}_n$ can be partitioned into symmetric chains.

Hint: For any symmetric chain $S_k, \ldots, S_{n-k}$ in $\mathcal{B}_n$ consider the following two (possibly empty) chains in $\mathcal{B}_{n+1}$: $S_{k+1}, \ldots, S_{n-k}$ and $S_k, S_k \cup \{n+1\}, \ldots, S_{n-k} \cup \{n+1\}$.

(c) Deduce Sperner's theorem from part (b).

Exercise 5 Let $x_1, \ldots, x_{2n}$ be real numbers with $|x_i| \geq 1$ for all i, and let $I \subset \mathbb{R}$ be an arbitrary open interval of length 2.

(a) Prove that the number of sums $\sum_{k=1}^{2n} \epsilon_k x_k$, where $\epsilon_k \in \{-1, +1\}$, which fall in the interior of I does not exceed $\binom{2n}{n}$.

(b) Show that for a closed interval I of length 2 the statement is not necessarily true.