Discrete Mathematics I So 18
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Exercise Sheet 8

Due date: June 12 at 4:15 PM

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 Prove that in any partial order on a set P of $n \geq sr + 1$ elements, where s, r are non-negative integers, there exists a chain of length $s + 1$ or an antichain of size $r + 1$.

Exercise 2 Let $a_1, \ldots, a_n$ be a sequence of n distinct positive integers such that $n \geq sr+1$. Use Exercise 1 to show that:

(a) we can either find a subset of A of $\{a_1, \ldots, a_n\}$ with $|A| = s + 1$ such that for every two elements in A one element divides the other, or a subset B with $|B| = r + 1$ such that no element of B divides the other;

(b) in this sequence either there is an increasing subsequence of $s + 1$ terms or a decreasing subsequence of $r + 1$ terms.

Exercise 3 Prove that the Möbius function of the poset $([n], \cdot|\cdot)$ is

$$\mu_{y,x} = \begin{cases} 1 & \text{if } \frac{x}{y} \text{ is squarefree and is the product of an even number of primes,} \\ -1 & \text{if } \frac{x}{y} \text{ is squarefree and is the product of an odd number of primes,} \\ 0 & \text{otherwise.} \end{cases}$$

[Hint (to be read backwards): $\frac{x}{y}$ no noitcudni usu]

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Exercise 4 Using the Möbius inversion formula and the statement from Exercise 3 prove the formula for the Euler totient function, namely that if the prime factroization of $n \in \mathbb{N}$ is $n = p_1^{\alpha_1} \cdots p_r^{\alpha_a}$ then we have $\varphi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)$.

[Hint (to be read backwards): $n = \sum_{d|n} \varphi(d)$ gnitats meroeht s'ssuaG esu]

Exercise 5 Let $(P, \leq)$ be a poset, and μ its Möbius function. Prove that whenever $x, y \in P$ are such that $y \leq x$, $x \neq y$ then $\mu_{y,x} = -\sum_{y<z\leq x} \mu_{z,x}$. [1]

[1] In the lecture we used the identity $\mu_{y,x} = -\sum_{y\leq z<x} \mu_{y,z}$ to define the Möbius function.

[Hint (to be read backwards): .rehto hcae fo sesrevni era ,tesop eht fo noitaler ytilibarapmoc eht dna noitcnuf suiböM eht gnitneserper ,Z dna M secirtam eht taht devorp ew erutcel eht ni taht tcaf eht esu]