Discrete Mathematics I So 18
Tibor Szabó
Tamás Mészáros
Anurag Bishnoi

Exercise Sheet 9

Due date: June 26 at 4:15 PM

You should try to solve all of the exercises below, but clearly mark which two solutions you would like us to grade – each problem is worth 10 points[1]. We encourage you to submit in pairs, but please remember to indicate the author of each solution.

Exercise 1 The n-dimensional integer lattice $\mathbb{Z}^n$ is the set of all the points in $\mathbb{R}^n$ that have integer coordinates.

(1) Prove that for any set of $2^n + 1$ points in $\mathbb{Z}^n$, there exist two points whose midpoint (in $\mathbb{R}^n$) is also contained in the integer lattice $\mathbb{Z}^n$.

(2) Show that this is the best we can do, by giving a set of 2^n points in $\mathbb{Z}^n$, for every $n \geq 1$, such that the midpoint of any two points in the set lies outside the integer lattice.

Exercise 2 We would like to draw a graph G while minimizing the number of times we have to lift our pen. However, after you draw an edge, you are not allowed to retrace the edge again, so that every edge of G should be drawn exactly once. Let's say when you first put your pen on paper to when you next take it off counts as one stroke *stroke*.

(1) Prove that is G has $2k$ vertices of odd degree, then at least k strokes are necessary to draw G.

(2) Show that any connected graph with exactly $2k$ vertices of odd degree can be drawn with k strokes.

Exercise 3 Let G be a k-regular graph of girth $g \geq 3$, i.e., the shortest cycle in G is of length g.

(1) Prove that if g is odd then the number of vertices in G is at least

$$1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2},$$

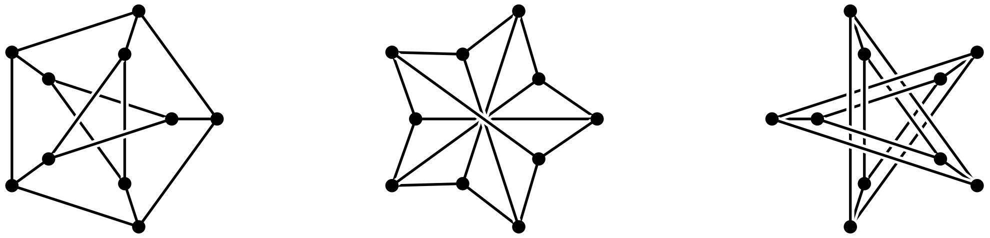
and if g is even then the number of vertices in G is at least

$$2(1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{(g-2)/2}).$$

(2) Describe the graphs meeting these two bounds for $g = 3$ and 4, respectively.

[1] unless stated otherwise

Exercise 4 Determine which pairs of graphs below are isomorphic.



Exercise 5 Show that a graph G is bipartite if and only if each subgraph H of G contains an independent set of size at least $|V(H)|/2$.

Exercise 6 [14 *points*] In the practice exam, you proved that the Ramsey number $R(4)$ is at most 32. Tweak that proof to show that $R(4) \leq 18$. If you are able to prove that $R(4) \leq k$ for any $18 \leq k \leq 31$, then you will get $32 - k$ points.