ELEMENTARY COUNTING PRINCIPLES

- RULE OF SUM: $S = \bigcup_{i=1}^{s} S_i \implies |S| = \sum_{i=1}^{s} |S_i|$

S_i are pairwise disjoint

i.e., $\forall i \neq j \; S_i \cap S_j = \cancel{\neq}\emptyset$

- Basis of every CASE ANALYSIS (Classify elements according to some property. Cases should be disjoint, cover everything)

Example: 1st grader "word problem"

Drawer with 8 pairs of yellow socks
5 -"- blue socks
3 -"- green socks

and no more.
How many socks are in the drawer?

Example: # of k-element subsets of an n-element set
$k, n \in \mathbb{N}_0$

Notation: $[n] := \{1, 2, \dots, n\}$ $\quad n \in \mathbb{N}$

X set
$k \in \mathbb{N}_0$ $\quad \binom{X}{k} := \{K \subseteq X : |K| = k\}$

$\binom{0}{0} := 1 = $ # of subsets of $\emptyset$

$\binom{n}{k} := \left|\binom{[n]}{k}\right| = $ # of k-element subsets of $[n]$

Example: $\binom{[4]}{3} = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$

Proposition (Pascal recurrence)

$\forall \; n \geq k \geq 1$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Remark: We do not "know" a formula for $\binom{n}{k}$ yet ...

Pf: Classify ~~#~~ k-subsets of $[n]$ according to whether they contain n or not.

$$S_1 = \left\{ T \in \binom{[n]}{k} : n \in T \right\}$$

$$S_2 = \left\{ T \in \binom{[n]}{k} : n \notin T \right\}$$

- $S_1 \cap S_2 = \emptyset$
- $S_1 \cup S_2 = \binom{[n]}{k}$

Sum Rule $\Rightarrow$ $\binom{n}{k} = \left|\binom{[n]}{k}\right| = |S_1| + |S_2|$

$$|S_2| = \left|\binom{[n-1]}{k}\right| = \binom{n-1}{k}$$

Bijection

$$S_1 \longleftrightarrow \binom{[n-1]}{k-1} \Rightarrow |S_1| = \left|\binom{[n-1]}{k-1}\right| = \binom{n-1}{k-1}$$

$$T \in S_1 \longleftrightarrow T \setminus \{n\} \in \binom{[n-1]}{k-1}$$

$\square$

Pascal triangle: Values can be calculated using recurrence and $\binom{n}{0} = \binom{n}{n} = 1 \quad \forall n \in \mathbb{N}_0$

1

1 1

1 2 1 Column k

1 3 3 1

1 4 6 4 1

1 ... $\binom{n-1}{k-1}$ $\binom{n-1}{k}$ ... 6+4=10 ... 1

Row n: 1 ... $\binom{n}{k}$ ... 1

<u>Notation:</u> product of sets $S_1, \dots, S_k$:

$$S_1 \times \dots \times S_k = \bigtimes_{i=1}^{k} S_i := \{(a_1, \dots, a_k) : a_i \in S_i \ \forall i=1,\dots,k\}$$

$$S^k := \underbrace{S \times \dots \times S}_{k\text{-times}}$$

$\boxed{\text{Rule of Product}}$ If a set $T = \bigtimes_{i=1}^{t} T_i$ is the product of some sets $T_1, \dots, T_t$, then

$$|T| = \prod_{i=1}^{t} |T_i|$$

Remark
This is the "efficient" Rule of Sum: if we classified the elements of T according to some mutually exclusive properties $e_1, \dots, e_s$ and the sets $S_i = \{x \in T : x \text{ has } e_i\}$ HAPPEN TO HAVE the same cardinality $|S_i| = |S_1| \ \forall i=1,\dots,s$

then $|T| = |S_1| + \dots + |S_s| = s \cdot |S_1|$

Pf: Rule of Sum (Classify according to first coordinate) + Induction on t

Corollary: $\boxed{|S^t| = |S|^t}$

<u>Example</u> (2nd grader) If the # of yellow, blue, and green sock in the drawer is each 8, then the socks form a product set: $\{Y, B, G\} \times [8]$ (and their # $3 \cdot 8$)

<u>Example</u> # of bitstrings of length n is 2^n

$\updownarrow$
0/1-sequences

$$|\{(a_1, \dots, a_n) : a_i \in \{0,1\} \ \forall i\}| = |\{0,1\}^n| = |\{0,1\}|^n = \underline{\underline{2^n}}$$

Example: Permutations

Def: For set X and $k \in \mathbb{N}$

- a *k-permutation of X* is an injective function $\sigma : [k] \longrightarrow X$

  (that is, a vector $(\sigma(1), \ldots, \sigma(k)) \in X^k$ with DISTINCT coordinates)

- a *permutation of X* is just an $|X|$-permutation of X

Notation: $X^{\underline{k}} := \{(a_1, \ldots, a_k) \in X^k : a_i \neq a_j \ \forall i \neq j\}$ = set of k-permutations of X

For $n \in \mathbb{R}$, $k \in \mathbb{N}_0$, the *k^{th} falling factorial* is $n^{\underline{k}} = \prod_{i=0}^{k-1} (n-i)$

Note: $n^{\underline{0}} = 1$

Proposition: $\forall\, n, k \in \mathbb{N}$ $\quad \left| [n]^{\underline{k}} \right| = n^{\underline{k}}$

Pf: For $k > n$ $\quad [n]^{\underline{k}} = \emptyset$ and $n^{\underline{k}} = n \cdot (n-1) \cdot \ldots \cdot (n-n) \cdot \ldots \cdot (n-k+1) = 0$

Let $1 \leq k \leq n$. Choose the coordinates of a k-permutation one after the other, starting with the first one:

$$(\bullet, \bullet, \bullet, \text{-----------}, \bullet) \in [n]^{\underline{k}}$$

- can be chosen n ways
- can be chosen $(n-1)$ ways (INDEPENDENT OF what was chosen as first coordinate
- can be chosen $(n-2)$ ways (INDEPENDENT OF what was chosen as first and second coordinate
- can be chosen $(n-k+1)$ ways (INDEPENDENT OF what was chosen before)

\# of choices

$$n \cdot (n-1) \cdot (n-2) \cdot \text{------} \cdot (n-k+1) = \prod_{i=0}^{k-1} (n-i) = n^{\underline{k}}$$

$\square$

Remark: $[n]^k$ is a set of vectors, but NOT a product of sets: the SET of elements which are possible as i^{th} coordinate depends on the first $(i-1)$ coordinates.

However the NUMBER of these elements does not!

GENERALIZED RULE OF PRODUCT Let $t_1, \dots, t_k \in \mathbb{N}$.

Let the elements of the set T be encoded as k-Tuples of answers for a sequence of k questions such that $\forall i=1,\dots,k$, given any valid sequence of answers to the first $(i-1)$ questions, the # of possible answers to Question i is t_i

(irrespective of what the first $(i-1)$ answers were. The set of answers and even the Questions CAN depend on the previous answers.)

Then $$|T| = \prod_{i=1}^{k} t_i$$

Pf: Rule of Sum (Classify according to answer to Question 1) + Induction on k.

Special Case: Rule of Product

$$T = T_1 \times \dots \times T_k$$

Question i: What is coordinate i?

$|T_i|$ possible answers

Rule of Bijection: If $\exists$ a bijection $F: S \to T \Rightarrow |S| = |T|$

Remark: • A well-chosen bijection can reveal how to enumerate an otherwise complicated-looking set.

• Even if the cardinality of two sets is known to be equal by algebraic (or other) means a combinatorial bijection can add much insight.

Example ① We had one already:

$$S_1 = \left\{ T : T \in \binom{[n]}{k},\ n \in T \right\} \xrightarrow{F} \binom{[n-1]}{k-1}$$

$$T \longrightarrow T \setminus \{n\}$$

Example ② # of subsets of $[n]$?

Power set of X $\quad 2^X := \{ T : T \subseteq X \}$ $\quad$ (Notation $\mathcal{P}(X)$ also used)

Proposition: $\left| 2^{[n]} \right| = 2^n \quad \forall n \in \mathbb{N}_0$

Pf: Bijection $F: 2^{[n]} \longrightarrow \{0,1\}^n$

$$\begin{array}{ccc} 2^{[n]} & \longrightarrow & \{0,1\}^n \\ \Downarrow & & \Downarrow \\ T & \longrightarrow & F(T) = (\ldots\ldots, F(T)_i, \ldots) \end{array}$$

$$F(T)_i = \begin{cases} 0 & \text{if } i \notin T \\ 1 & \text{if } i \in T \end{cases}$$

• F is injective: $T \neq T' \Rightarrow \exists i \in (T \setminus T') \cup (T' \setminus T)$

$\Rightarrow F(T)_i \neq F(T')_i$

• F is surjective: For $\forall v \in \{0,1\}^n$ $\exists$ set $T \subseteq [n]$ s.t. $F(T) = v$, namely $T := \{ i \in [n] : v_i = 1 \}$.

So $\left| 2^{[n]} \right| \underset{\text{Bijection}}{=} \left| \{0,1\}^n \right| \underset{\text{Product}}{=} |\{0,1\}|^n = 2^n$

$\square$

Example 3. What is $\binom{n}{k}$?

Bijection

$$[n]^{\underline{k}} \xrightarrow{\;F\;} \binom{[n]}{k} \times [k]^{\underline{k}}$$

$$(a_1, \dots, a_k) \longmapsto \left(\{a_1, \dots, a_k\}, \pi\right)$$

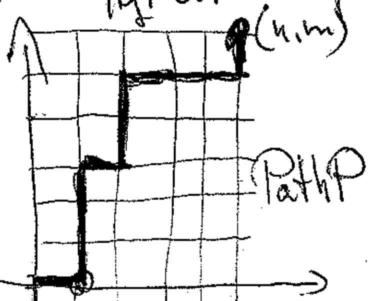
$\pi : [k] \to [k]$ is the (unique) permutation such that $a_{\pi(1)} < a_{\pi(2)} < \dots < a_{\pi(k)}$

- F is injection [both $\{a_1, \dots, a_k\}$ and π are uniquely determined by $(a_1, \dots, a_k)$]
- F is surjection [Given $(\{b_1, \dots, b_k\}, \sigma)$, it is equal to $F((b_{\sigma^{-1}(1)}, \dots, b_{\sigma^{-1}(k)}))$]

$$\Rightarrow \; n^{\underline{k}} = \left|[n]^{\underline{k}}\right| = \left|\binom{[n]}{k} \times [k]^{\underline{k}}\right| = \left|\binom{[n]}{k}\right| \cdot \left|[k]^{\underline{k}}\right| = \binom{n}{k} \cdot k!$$

$$\Rightarrow \; \binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

Example 4. # nondecreasing lattice paths from $(0,0)$ to (n,m) $=: L(n,m) = \binom{n+m}{n}$



Encoding

$$\text{Path } P \longrightarrow \underbrace{\text{RUUURUURRRU}}_{w} \longrightarrow \cancel{\binom{[n+m]}{n}}$$

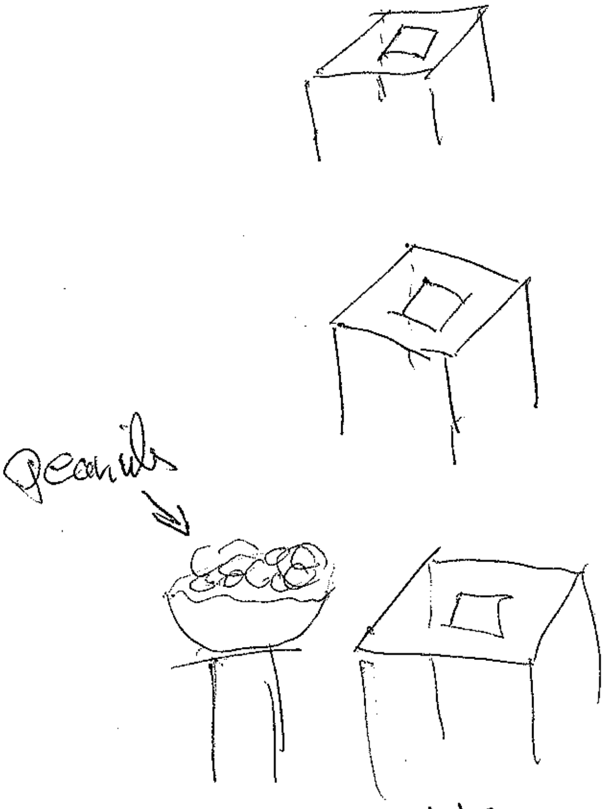
$\{i : w_i = R\}$

$$L(n,m) \xrightarrow{\text{Bijection}} \left\{\begin{array}{l} \text{R/U words with} \\ n \text{ Rs and} \\ m \text{ Us} \end{array}\right\} \longrightarrow \binom{[n+m]}{n}$$

- $F(\text{Path}) \in \binom{[n+m]}{n}$ since $\forall$ Path has exactly n Right moves exactly m Up moves
- F is injection ($P \neq P' \Rightarrow F(P)$ differs from $F(P')$ in the element corresponding to the first time P separates from P')
- F is surjection (n Right move and m Up move in ANY order takes a path from $(0,0)$ to (n,m))

<u>EXAMPLE</u>: The importance of knowing/identifying what exactly we WANT to count.

6 people: How many ways are there to seat them to play chess at 3 boards?



FORMULATE A PRECISE QUESTION!!!

- Does it matter who plays black or white?

- Does it matter who sits next to the peanuts?

For us: NO and NO (HW: Solve other instances)

So we enumerate the set $\left\{\{A_1, A_2, A_3\} : \begin{array}{l} A_1 \cup A_2 \cup A_3 = [6] \\ |A_1| = |A_2| = |A_3| = 2 \end{array}\right\}$

- Choose A_1 $\binom{6}{2}$ ways
- Once A_1 is chosen $\exists \left|\binom{[6] \setminus A_1}{2}\right| = \binom{4}{2}$ ways to choose A_2
- A_1 and A_2 are chosen $\Rightarrow$ $A_3 = [6] \setminus (A_1 \cup A_2)$ <u>unique</u>

Every <u>set</u> $\{A_1, A_2, A_3\}$ was counted $3! = 6$ ways this way.

So: $\dfrac{\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2}}{3!} = \dfrac{15 \cdot 6 \cdot 1}{6} = \underline{\underline{15}}$

Another way: Alice, Bob, Carole, Daniel, Emma, Frank

Question #1: Who is the <u>partner</u> of Alice?

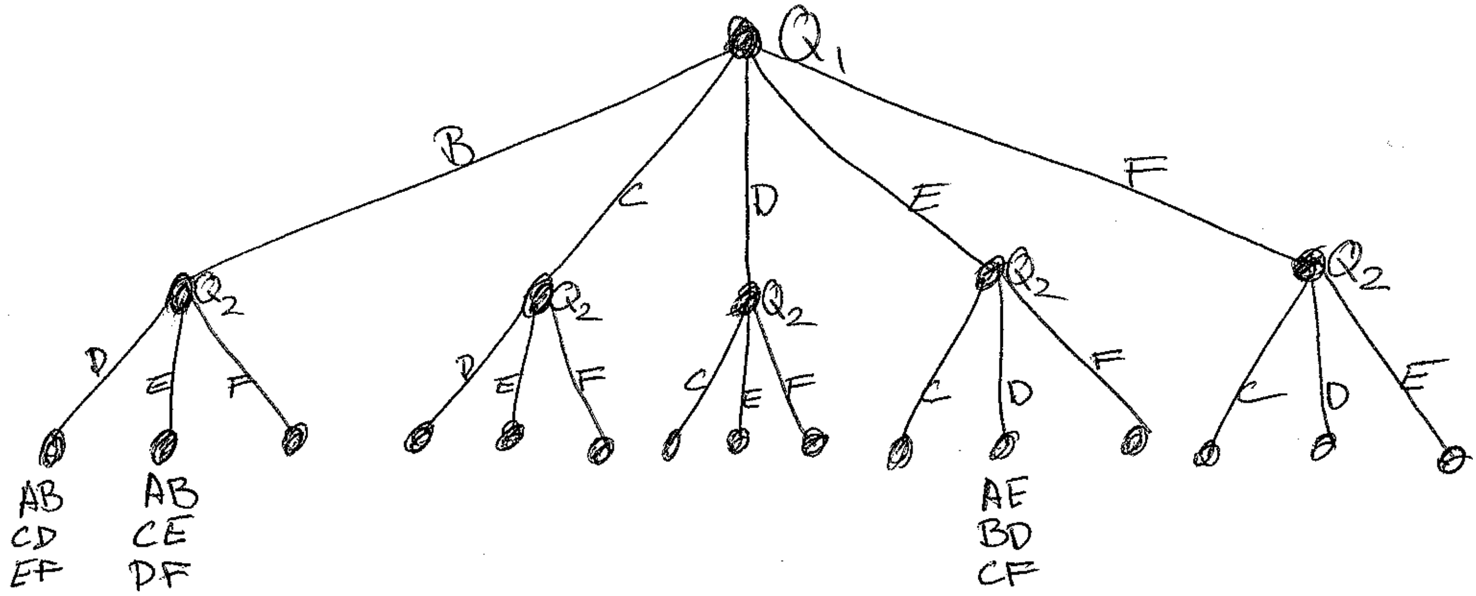
5 possible answers

Question #2: Who is the partner of the person whose name starts with the letter coming first in the alphabet, ~~among these~~ after removing Alice and her partner from consideration?

3 possible answers – independent of the first answer

(**Question #3:** ~~Who is the~~ ....

1 possible answer unique pair remains)



So $5 \cdot 3 = 15$ ways, different answers lead to different pairings

(Same answer as for first solution, phew....)

In general, to distribute $n=2k$ people into k pairs

Two solutions $\rightsquigarrow$ identity

$$\frac{\binom{n}{2}\cdot\binom{n-2}{2}\cdot\ldots\cdot\binom{2}{2}}{\left(\frac{n}{2}\right)!} = \overbrace{(n-1)(n-3)\cdot\ldots\cdot(n-(2i+1))\cdot\ldots\cdot 3\cdot 1}^{(n-1)!!}$$

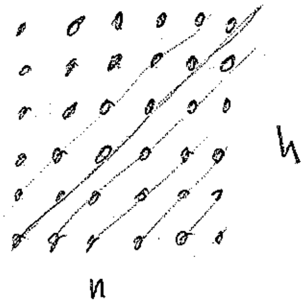
Question #i: Who is the partner of the person whose name comes first in the alphabet among ~~who~~ those who are not paired yet?

Key: While the SET of possible answers MIGHT depend on the answers to the first $i-1$ questions, the NUMBER of possible answers does NOT!

Double Counting (Rule of counting two ways)

- When two formulas enumerate the same set they must be equal.
- Exchange of summation (Finite Fubini Thm.)

Count gridpoints in



(1) vertical line by vertical line $n \cdot n = n^2$

(2) diagonal by diagonal

$$1+2+3+4+\dots+n-1+n+n-1+n-2+\dots+2+1$$

$$= 2(1+2+\dots+n-1)+n$$

$$\Rightarrow \quad n^2 = 2\sum_{i=1}^{n-1} i + n$$

$$\Rightarrow \binom{n}{2} = \frac{n^2-n}{2} = \sum_{i=1}^{n-1} i$$

Example: Number theoretic fn $d : \mathbb{N} \to \mathbb{N}$

$d(n) = \#$ of divisors of n

Real time exercise: Evaluate $d(n)$ for $1 \le n \le 8$

$d(1) = 1$

$d(2) = 2$

$d(3) = 2$ $\qquad d(8191) = 2$

$d(4) = 3$ $\qquad d(8192) = 14$

$d(5) = 2$

$d(6) = 4$

$d(7) = 2$

$d(8) = 4$ $\qquad$ d jumps up and down

What is the average value?

$$\bar{d}(n) = \frac{\sum_{i=1}^{n} d(i)}{n}$$

$\bar{d}(1) = 1$

$\bar{d}(2) = \frac{3}{2}$

$\bar{d}(3) = \frac{5}{3}$

$\bar{d}(4) = 2$

$\bar{d}(5) = 2$

$\bar{d}(6) = \frac{7}{3}$

$\bar{d}(7) = \frac{16}{7}$

$\bar{d}(8) = \frac{5}{2}$

Double counting to the rescue

$$\sum_{i=1}^{n} d(i) = \sum_{i=1}^{n} \sum_{\substack{1\le j\le n\\ j\mid i}} 1 = \left|\left\{(j,i) : i\in[n],\ j\in[n],\ j\mid i\right\}\right|$$

$$= \sum_{j=1}^{n} \underbrace{\sum_{\substack{1\le i\le n\\ j\mid i}} 1}_{\text{In this sum } j \text{ is fixed and we count the \# multiples of } j \text{ up to } n} = \sum_{j=1}^{n} \left\lfloor \frac{n}{j} \right\rfloor$$

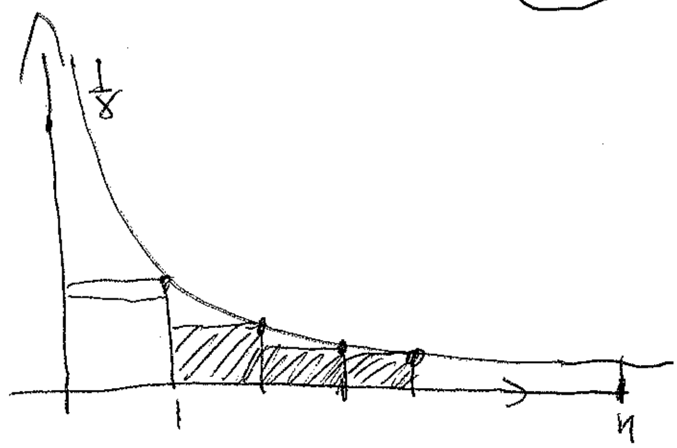
Exchange summation

In this sum j is fixed and we count the # multiples of j up to n

Estimate

$$n(H_n - 1) = \sum_{j=1}^{n}\left(\frac{n}{j} - 1\right) \le \sum_{j=1}^{n} \left\lfloor \frac{n}{j} \right\rfloor \le \sum_{j=1}^{n} \frac{n}{j} = n \underbrace{\sum_{j=1}^{n} \frac{1}{j}}_{H_n}$$

H_n Harmonic number



$$H_{n-1} = \sum_{j=1}^{n-1} \frac{1}{j} \ge \underbrace{\int_1^n \frac{1}{x}\,dx}_{\ln n} \ge \sum_{j=2}^{n} \frac{1}{j} = H_n - 1$$

So

$$H_n - 1 \le \frac{n(H_n-1)}{n} \le \overbrace{\frac{\sum_{i=1}^{n} d(i)}{n}}^{\bar{d}(n)} \le \frac{nH_n}{n} = H_n$$

Binomial Identities

- $\binom{n}{k} = \binom{n}{n-k}$ Pf: Bijection $A \xrightarrow{F} \bar{A}$

$$A \in \binom{[n]}{k} \qquad \bar{A} \in \binom{[n]}{n-k}$$

Sum of a row in Pascal's Δ

- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ Pf: Sum Rule: Classify subsets of $[n]$ according to size

$$\bigcup_{k=0}^{n} \binom{[n]}{k} = 2^{[n]}$$

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \left|\binom{[n]}{k}\right| = \left|\bigcup \binom{[n]}{k}\right| = \left|2^{[n]}\right| = 2^n$$

Binomial Thm:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Pf: Induction on n (and recurrence)

Applications: $x = y = 1 \Rightarrow 2^n = (1+1)^n = \sum \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$

$x = 1,\ y = -1$ $\quad 0^n = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k (-1)^{n-k} = \sum_{j=0}^{n} (-1)^{j} \binom{n}{j}$

$$\Rightarrow \sum_{\substack{0 \le j \le n \\ \text{even}}} \binom{n}{j} = \sum_{\substack{0 \le j \le n \\ j \text{ odd}}} \binom{n}{j}$$

\# of subsets of even size = \# of subsets of odd size

UNLESS $n = 0$ ($0^0 = 1$, $\emptyset$ has 1 subset)

Alternative combinatorial proof:

$$\mathcal{O} = \{T \subseteq [n] : |T| \text{ odd}\}$$

$$\mathcal{E} = \{T \subseteq [n] : |T| \text{ even}\}$$

$$\begin{array}{ccc} A & \xrightarrow{F} & \overline{A} \\ \cap & & \cap \\ \mathcal{O} & & \mathcal{E} \end{array}$$

works if n is odd (F is bijection)

In general:

$$A \longrightarrow \begin{cases} A \setminus \{n\} & \text{if } n \notin A \\ A \cup \{n\} & \text{if } n \in A \end{cases}$$

$$\begin{array}{ccc} \cap & & \cap \\ \mathcal{O} & & \mathcal{E} \end{array}$$

bijection $\Longrightarrow |\mathcal{O}| = |\mathcal{E}|$

~~Plan~~ Substitute $x=2$ $y=1$ $\Rightarrow$ $3^n = (2+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k$

<u>Combinatorial Proof:</u>

Meaning of RHS: $\binom{n}{k} \cdot 2^k = \left| \left\{ (K,S) : K \in \binom{[n]}{k},\ S \subseteq K \right\} \right|$

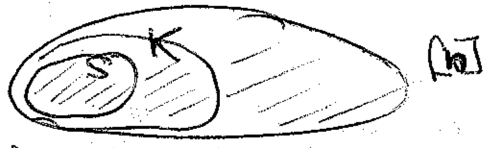
$\binom{n}{k}$: choose k-subset K of $[n]$

2^k: choose a subset of K

$$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = \left| \biguplus_{k} \left\{ (K,S) : K \in \binom{[n]}{k},\ S \subseteq K \right\} \right|$$

$$= \left| \left\{ (K,S) : K \in 2^{[n]},\ S \subseteq K \right\} \right|$$

Classified according to size of first coordinate



Partition von $[n] = A_1 \uplus A_2 \uplus A_3$

$$\left\{ (K,S) : K \in 2^{[n]},\ S \subseteq K \right\} \xrightarrow{F} \left\{ (A_1, A_2, A_3) : A_1 \uplus A_2 \uplus A_3 = [n] \right\}$$

$$(K,S) \longmapsto (S,\ K \setminus S,\ [n] \setminus K)$$

F is bijection:

- injection: $(K_1, S_1) \neq (K_2, S_2)$ wenn $K_1 \neq K_2 \Rightarrow [n] \setminus K_1 \neq [n] \setminus K_2$
  wenn $S_1 \neq S_2 \Rightarrow$ ~~S_1~~
- surjection: $(A_1, A_2, A_3) = F(A_1,\ A_1 \cup A_2)$

$$\left\{ (A_1, A_2, A_3) : A_1 \uplus A_2 \uplus A_3 = [n] \right\} \longrightarrow \{1,2,3\}^{[n]}$$

Bijection $\qquad (A_1, A_2, A_3) \longmapsto (v_1, \ldots, v_n)$ s.t.

if $i \in A_j$ then $v_i = j$

Notation $X^Y = \{ f : Y \to X \}$. Wenn $Y = [n]$ $X^{[n]}$