Möbius inversion

<u>Aim:</u> Generalize the inclusion-exclusion principle.

<u>Examples:</u>

1. $(a_1, a_2, \ldots\ldots)$, $a_i \in \mathbb{R} \longrightarrow (b_1, b_2, \ldots)$ $b_i = \sum_{j=0}^{i} a_i$

Backwards formula: $a_1 = b_1$, $\forall n \geq 2$ $a_n = b_n - b_{n-1}$

2. Let $f : 2^{[n]} \longrightarrow \mathbb{R}$ $\rightsquigarrow$ Define: $g : 2^{[n]} \longrightarrow \mathbb{R}$ by

$$g(T) = \sum_{S \subseteq T} f(S)$$

Can we express $f(S)$ in terms of g?

3. Let $f : \mathbb{N} \longrightarrow \mathbb{R}$ $\rightsquigarrow$ Define: $g : \mathbb{N} \longrightarrow \mathbb{R}$ by

$$g(n) = \sum_{d \mid n} f(d)$$

Can we express $f(n)$ in terms of g?

All these examples follow the same scheme:

$(P, \leq)$ poset, $f : P \longrightarrow \mathbb{R}$ function

Define $g : P \longrightarrow \mathbb{R}$ by $g(y) = \sum_{x \leq y} f(x)$ $\forall y \in P$

<u>Question:</u> What is f in terms of g?

<u>Examples</u>

1. $(\mathbb{N}_1, \leq)$
2. $(2^{[n]}, \subseteq)$
3. $(\mathbb{N}, \cdot \mid \cdot)$

Def A poset $(P, \leq)$ is <u>locally finite</u> if $\forall a, b \in P$ the <u>interval</u> $[a,b] = \{x \in P : a \leq x \leq b\}$ is finite.

– all examples before were locally finite

<u>Thm</u> (Möbius inversion formula)

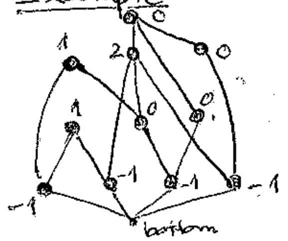
Let $(P, \leq)$ be a finite poset, and suppose we have a function $f : P \to R$. If $g : P \to \mathbb{R}$ is defined by $g(y) = \sum_{x \leq y} f(x)$, then

$$f(y) = \sum_{x \leq y} g(x) \cdot \mu_{x,y}$$

, where μ, the <u>Möbius function</u> of the poset, is given by

$$\mu_{xy} = \begin{cases} 0 & \text{if } x \not\leq y, \\ 1 & \text{if } x = y, \text{ and} \\ -\sum_{x \leq z < y} \mu_{xz} & \text{if } x < y. \end{cases}$$

<u>Example</u>



<u>Remark</u> We assume $(P, \leq)$ is finite. This is not necessary, the thm holds in a more general setting.

Then if $(P, \leq)$ is locally finite, $f : P \to \mathbb{R}$, $x_0 \in P$ fixed and

inversion from below

inversion from above

– if $g(y) = \sum_{x_0 \leq x \leq y} f(x) \leadsto f(y) = \sum_{x_0 \leq x \leq y} g(x) \cdot \mu_{x,y}$

– if $g(y) = \sum_{y \leq x \leq x_0} f(x) \leadsto f(y) = \sum_{y \leq x \leq x_0} g(x) \cdot \mu_{y,x}$

I. Möbius inversion

<u>Aim:</u> Generalize the inclusion - exclusion principle.

<u>Examples:</u>

(1.) $(a_1, a_2, \ldots\ldots),\ a_i \in \mathbb{R} \longrightarrow (b_1, b_2, \ldots)\quad b_i = \sum_{j=0}^{i} a_i$

Backwards formula: $a_1 = b_1$, $\forall n \geq 2\ \ a_n = b_n - b_{n-1}$

(2.) Let $f : 2^{[n]} \longrightarrow \mathbb{R} \rightsquigarrow$ Define: $g : 2^{[n]} \longrightarrow \mathbb{R}$ by

$$g(T) = \sum_{S \subseteq T} f(S)$$

Can we express $f(S)$ in terms of g?

(3.) Let $f : \mathbb{N} \longrightarrow \mathbb{R} \rightsquigarrow$ Define: $g : \mathbb{N} \longrightarrow \mathbb{R}$ by

$$g(n) = \sum_{d/n} f(d)$$

Can we express $f(n)$ in terms of g?

All these examples follow the same scheme:

$(P, \leq)$ poset, $f : P \longrightarrow \mathbb{R}$ function

Define $g : P \longrightarrow \mathbb{R}$ by $g(y) = \sum_{x \leq y} f(x) \quad \forall y \in P$

<u>Question:</u> What is f in terms of g?

<u>Examples</u>

(1.) $(\mathbb{N}_1, \leq)$ (2.) $(2^{[n]}, \subseteq)$ (3.) $(\mathbb{N}, \cdot | \cdot)$

$$= \sum_{i\le k\le j} \mathbb{1}_{[x_k\le x_j]}\,\mu_{x_i x_k} = \sum_{x_i\le z\le x_j} \mu_{x_i z} =$$

$$= \underbrace{\sum_{x_i\le z< x_j} \mu_{x_i z}}_{-\mu_{x_i x_j}} + \mu_{x_i x_j} = \begin{cases} 0 & \text{if } i\ne j \\ 1 & \text{if } i=j \end{cases}$$

By Claim: $\bar{f} = \bar{g} M \iff \forall y \;\; f(y) = \sum_{x\le y} g(x)\mu_{xy}$ ■

<u>Examples</u>

1. <u>Prop</u> $([n], \le)$ $\rightsquigarrow$ The Möbius function is

$$\mu_{xy} = \begin{cases} 1 & \text{if } y = x, \\ -1 & \text{if } y = x+1, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \qquad \forall x, y \in [n]$$

<u>Proof</u> Induction on $y - x$.

If $y - x < 0 \implies x \not\le y \implies \mu_{xy} = 0$

If $y - x = 0 \implies x = y \implies \mu_{xx} = 1$

If $y - x = 1 \implies y = x+1 \implies$

$$\mu_{x,x+1} = -\sum_{x\le z< x+1} \mu_{xz} = -\mu_{xx} = -1$$

Now suppose $y - x \ge 2$.

$$\mu_{x,y} = -\sum_{x\le z<y} \mu_{xz} = -\Big(\underset{\substack{\parallel\\1}}{\mu_{xx}} + \underset{\substack{\parallel\\-1}}{\mu_{x,x+1}} + \sum_{x+2\le z<y} \mu_{xz}\Big) = 0$$

by induction $= 0$ ■

Corollary Let $f: \mathbb{N} \to \mathbb{R}$, and define $g: \mathbb{N} \to \mathbb{R}$ by $g(n) = \sum_{i=1}^{n} f(i)$, then $f(n) = g(n) - g(n-1)$ $\forall n \geq 2$.

Proof By Möbius inversion formula (applied to $([n], \leq)$):

$$f(n) = \sum_{i=1}^{n} g(i)\mu_{i,n} = g(n) - g(n-1). \quad \blacksquare$$

(2) Prop $(2^{[n]}, \subseteq) \rightsquigarrow$ The Möbius function is

$$\mu_{S,T} = \begin{cases} (-1)^{|T \setminus S|} & \text{if } S \subseteq T, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof If $S \not\subseteq T$, then by definition $\mu_{S,T} = 0$.
Now suppose $S \subseteq T$, and apply induction on $|T \setminus S|$

If $S = T \rightsquigarrow$ by def. $\mu_{SS} = 1 = (-1)^0$

If $|T \setminus S| = 1 \rightsquigarrow$ by def. $\mu_{S,T} = -\sum_{S \subseteq F \subsetneq T} \mu_{SF} = -\mu_{SS}$
$= (-1) = (-1)^1$.

Now suppose $|T \setminus S| = t \geq 2$.

$$\mu_{S,T} = -\sum_{S \subseteq F \subsetneq T} \mu_{S,F} = -\sum_{S \subseteq F \subsetneq T} (-1)^{|F \setminus S|} = -\sum_{i=0}^{t-1} \binom{t}{i} (-1)^i$$

$$= -\Bigg(\underbrace{\sum_{i=0}^{t} \binom{t}{i} (-1)^i \cdot 1^{t-i}}_{(-1+1)^t = 0} - \binom{t}{t}(-1)^t \Bigg) = (-1)^t \quad \blacksquare$$

Corollary Let $f: 2^{[n]} \to \mathbb{R}$, and define $g: 2^{[n]} \to \mathbb{R}$ by $g(T) = \sum_{S \subseteq T} f(S)$, then

$$f(T) = \sum_{S \subseteq T} (-1)^{|T \setminus S|} \cdot g(S)$$

Proof Apply Möbius inv. formula for $(2^{[n]}, \subseteq)$ ∎

Corollary (Inclusion-exclusion formula)

$A_1, A_2, \dots, A_n \subseteq X$ finite. Then

$$|X \setminus \bigcup_{i=1}^{n} A_i| = \sum_{I \subseteq [n]} (-1)^{I} \cdot \Big| \bigcap_{i \in I} A_i \Big|.$$

Proof We will apply Möbius inversion on $(2^{[n]}, \subseteq)$ and a cleverly chosen f:

$$f(T) := \Big| \underbrace{\Big(X \setminus \bigcup_{i \in T} A_i \Big) \cap \Big(\bigcap_{j \in [n] \setminus T} A_j \Big)}_{B_T} \Big|$$ These sets partition X.

= number of elements $x \in X$ s.t. $x \notin A_i \ \forall i \in T$ and $x \in A_j \ \forall j \in [n] \setminus T$.

Let

$$g(T) = \sum_{S \subseteq T} f(S) = \Big| \bigcup_{S \subseteq T} B_T \Big|$$ — number of elements $x \in X$ s.t. $x \in A_i$ $\forall i \in [n] \setminus T$

$$\Rightarrow g(T) = \Big| \bigcap_{i \in [n] \setminus T} A_i \Big|.$$

Applying the previous corollary for $f([n]) = |X \setminus \bigcup_{i=1}^{n} A_i|$ we get

$$f([n]) = \sum_{S \subseteq [n]} (-1)^{|[n] \setminus S|} \cdot g(S) =$$

$$= \sum_{S \subseteq [n]} (-1)^{|[n] \setminus S|} \cdot \left| \bigcap_{i \in [n] \setminus S} A_i \right| =$$

$I = [n] \setminus S$

$$= \sum_{I \subseteq [n]} (-1)^{|I|} \cdot \left| \bigcap_{i \in I} A_i \right| \quad \blacksquare$$