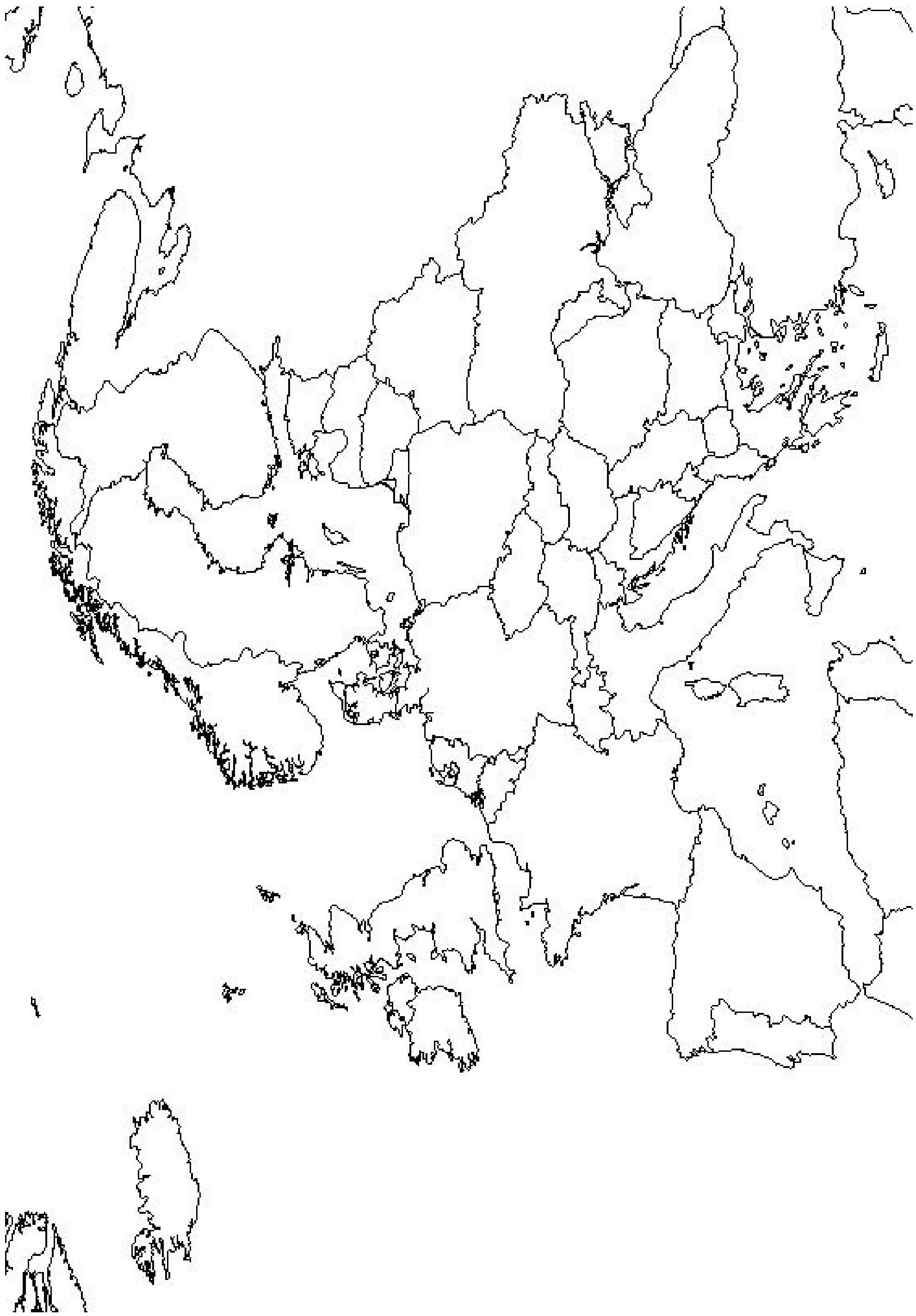
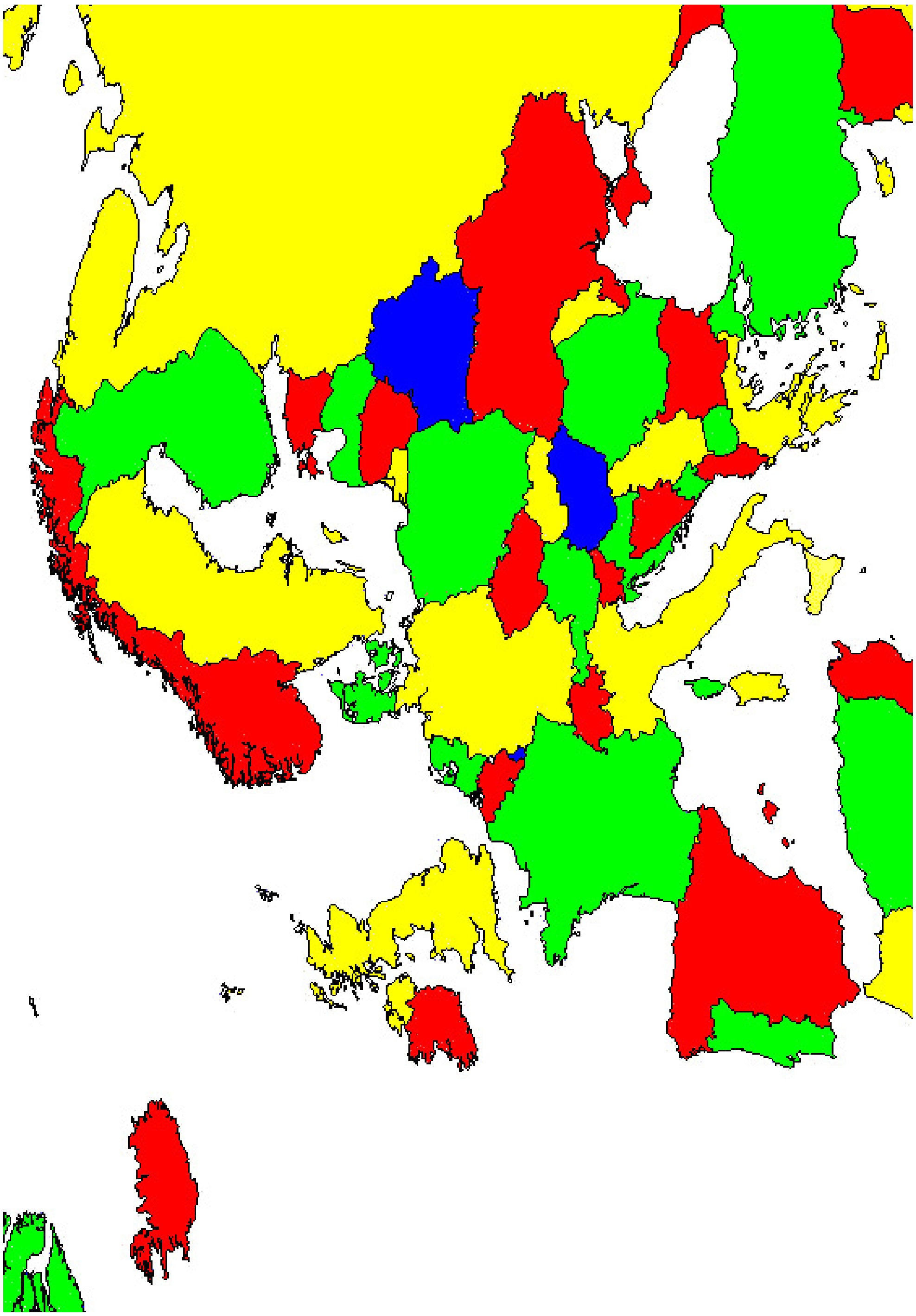
How many colors are needed to color a map?



Is 4 always enough?



Two relevant concepts

How many colors do we need to color a map so neighboring countries get different colors?
Simplifying assumption (not true in reality): Each country is bounded by a simple continuous curve.

Auxiliary graph: $V(G) =$ set of countries, $E(G) =$ pairs of countries that are neighboring (share a 1-dimensional piece of their boundary. (just points are not enough!)

Graph colorings: We then want a coloring of the *vertices* of this auxiliary graph, such that adjacent vertices receive distinct colors.

Planar graphs: The auxiliary graph G of the map has a special property: it can be drawn into the plane such that the edges do not cross. Indeed: draw the vertex representing the country in the "middle" (the "capitol") and draw a curve to the middle of the boundary curve with each country. This drawing forms an embedding of the graph G in the plane so that the edges do not intersect.

Vertex coloring, chromatic number

A k-coloring of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are called colors; the vertices of one color form a color class.

A k-coloring is proper if adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring.

The chromatic number is

$$\chi(G) := \min\{k : \ G \text{ is } k\text{-colorable}\}.$$

A graph G is k-chromatic if $\chi(G) = k$.

Examples. K_n, $K_{n,m}$, C_5, Petersen

A graph G is k-color-critical (or k-critical) if $\chi(H) < \chi(G) = k$ for every *proper* subgraph H of G.

Characterization of 1-, 2-, 3-critical graphs.

A simple lower bound and its tightness

Observation For every graph G, $\chi(G) \geq \omega(G)$.

Examples for $\chi(G) = \omega(G)$:

- cliques, bipartite graphs, complement of bipartite graphs (HW)
- *interval graphs* (scheduling)

  An interval representation of a graph is an assignment of an interval to the vertices of the graph, such that two vertices are adjacent iff the corresponding intervals intersect. A graph having such a representation is called an interval graph.

  Proposition. If G is an interval graph, then

  $$\chi(G) = \omega(G).$$

  Proof. Order vertices according to left endpoints of corresponding intervals and color *greedily*.

When $\chi(G) \neq \omega(G)$

Another lower bound. For every graph G, we have

$$\chi(G) \geq \frac{v(G)}{\alpha(G)}.$$

Proof. Each color class is an independent set.

Examples for $\chi(G) \neq \omega(G)$:

- odd cycles of length at least 5
  - $\alpha(C_{2k+1}) \leq k$ (count edges from an independent set to the rest)
  - then: $\chi(C_{2k+1}) \geq \frac{v(C_{2k+1})}{\alpha(C_{2k+1})} = 2 + \frac{1}{k} > 2$
  - **BUT:** $\omega(C_{2k+1}) = 2$, when $2k + 1 \geq 5$

- complement of odd cycles of length at least 5
  - $\omega(\overline{C}_{2k+1}) = \alpha(C_{2k+1}) = k$,
  - **BUT:** $\chi(\overline{C}_{2k+1}) \geq \frac{v(\overline{C}_{2k+1})}{\alpha(\overline{C}_{2k+1})} = k + \frac{1}{2} > k$ when $2k + 1 \geq 5$

Perfect graphs

Definition (Berge) A graph G is perfect, if $\chi(H) = \omega(H)$ for every induced subgraph $H \subseteq G$.

Conjectures of Berge (1960)

Strong Perfect Graph Conjecture. G is perfect iff G does not contain an induced subgraph isomorphic to an odd cycle of order at least 5 or the complement of an odd cycle of order at least 5.

Weak Perfect Graph Conjecture. G is perfect iff $\overline{G}$ is perfect.

The first conjecture was converted into the Weak Perfect Graph Theorem by Lovász (1972)

The second conjecture was converted into the Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour, Thomas (2002)

Mycielski's Construction

Q: How *bad* could the bound $\chi(G) \geq \omega(G)$ be?
A: Very bad; there are graphs with $\omega(G) = 2$
and $\chi(G) = 10^{10^{10}}$

Construction. Given graph G with vertices $v_1, \ldots, v_n$, we define supergraph $M(G)$:

$V(M(G)) = V(G) \cup \{u_1, \ldots u_n, w\}$.

$E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}\}$.

Theorem.
(i) If G is triangle-free, then so is $M(G)$.
(ii) If $\chi(G) = k$, then $\chi(M(G)) = k + 1$.

And it gets even worse ...

Typical behaviour of the lower bounds:
For the uniform random graph $G = G(n, \frac{1}{2})$,

$$\chi(G) \approx \frac{n}{2 \log_2 n}$$
$$\omega(G), \alpha(G) \approx 2 \log_2 n$$