Coloring maps with 5 colors

Six Color Theorem. If G is planar, then $\chi(G) \leq 6$.

Proof. By Euler, minimum degree is at most 5. Then

Proposition $\chi(G) \leq \max_{H \subseteq G} \delta(H) + 1$.

Proof. Greedy coloring procedure with the ordering $v_1, \ldots, v_n$, where v_i is a min-degree vertex of the graph $G[\{v_1, \ldots, v_n\}]$.

Five Color Theorem. (Heawood, 1890) If G is planar, then $\chi(G) \leq 5$.

Proof. Take a minimal counterexample.

(i) There is a vertex v of degree at most 5.

(ii) Modify a proper 5-coloring of $G - v$ to obtain a proper 5-coloring of G. A contradiction.
(*Idea of modification:* Kempe chains.)

Coloring maps with 4 colors

Four Color Theorem. (Appel-Haken, 1976) For any planar graph G, $\chi(G) \leq 4$.

Idea of the proof.
W.l.o.g. we can assume G is a planar triangulation.
A configuration in a planar triangulation is a separating cycle C (the ring) together with the portion of the graph inside C.
For the Four Color Problem, a set of configurations is an unavoidable set if a minimum counterexample must contain a member of it.
A configuration is reducible if a planar graph containing it cannot be a minimal counterexample.

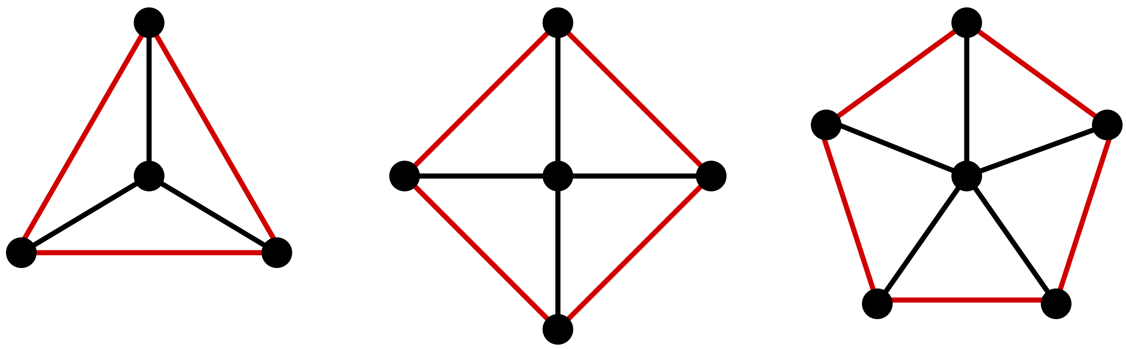
The usual proof attempts to

(i) find a set $\mathcal{C}$ of unavoidable configurations, and

(ii) show that each configuration in $\mathcal{C}$ is reducible.

Proof attempts of the Four Color Theorem

Kempe's original proof tried to show that the unavoidable set



is reducible.

Appel and Haken found an unavoidable set of 1936 of configurations, (all with ring size at most 14) and proved each of them is reducible. (1000 hours of computer time)

Robertson, Sanders, Seymour and Thomas (1996) used an unavoidable set of 633 configuration. They used 32 rules to prove that each of them is reducible. (3 hours computer time)

Kuratowski's Theorem

Theorem(Kuratowski, 1930) A graph G is planar iff G does not contain a subdivision of K_5 or $K_{3,3}$.

Proof.

A Kuratowski subgraph of G is a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. A minimal nonplanar graph is a nonplanar graph such that every proper subgraph is planar.

A counterexample to Kuratowski's Theorem constitutes a nonplanar graph that does not contain any Kuratowski subgraph.

Kuratowski's Theorem follows from the following Main Lemma and Theorem.

The spine of the proof

Main Lemma. If G is a graph with fewest edges among counterexamples, then G is 3-connected.

Lemma 1. Every minimal nonplanar graph is 2-connected.

Lemma 2. Let $S = \{x, y\}$ be a separating set of G. If G is a nonplanar graph, then adding the edge xy to some S-lobe of G yields a nonplanar graph.

Main Theorem.(Tutte, 1960) If G is a 3-connected graph with no Kuratowski subgraph, then G has a convex embedding in the plane with no three vertices on a line.

A convex embedding of a graph is a planar embedding in which each face boundary is a convex polygon.

Lemma 3. If G is a 3-connected graph with $n(G) \geq 5$, then there is an edge $e \in E(G)$ such that $G \cdot e$ is 3-connected.

Notation: $G \cdot e$ denotes the graph obtained from G after the contraction of edge e.

Lemma 4. G has no Kuratowski subgraph $\Rightarrow$ $G \cdot e$ has no Kuratowski subgraph.

Proof of Tutte's Theorem

Main Theorem. (Tutte, 1960) If G is a 3-connected graph with no Kuratowski subgraph, then G has a convex embedding in the plane with no three vertices on a line.

Proof. Induction on $n(G)$.
Base case: G is 3-connected, $n(G) = 4 \Rightarrow K_4$.

Let $e \in E$ s.t. $H = G \cdot e$ is 3-connected. (Lemma 3)
Then H has no Kuratowski subgraph. (Lemma 4)
Induction $\Rightarrow$ H has a convex embedding in the plane with no three vertices on a line.

Let $z \in V(H)$ be the contracted e.
$H - z$ is 2-connected $\Rightarrow$ boundary of the face containing z after the deletion of the edges incident to z is a cycle C.

Let $x_1, \ldots, x_k$ be the neighbors of x on C in cyclic order. Note that $|N(x)| \geq 3$ and hence $k \geq 2$.

Denote by $\langle x_i, x_{i+1} \rangle$ the portion of C from x_i to x_{i+1} (including endpoints; indices taken modulo k.)

Let $N_x = N(x) \setminus \{y\}$ and $N_y = N(y) \setminus \{x\}$.

Case 1. $|N_x \cap N_y| \geq 3$.

Let $u, v, w \in N_x \cap N_y$. Then x, y, u, v, w are the branch vertices of a K_5-subdivision in G.

Case 2. $|N_x \cap N_y| \leq 2$.

Since $|N_x \cup N_y| \geq 3$, there is w.l.o.g. a vertex $u \in N_y \setminus N_x$. Let i be such that u is on $\langle x_i, x_{i+1} \rangle$.

Case 2a. N_y is contained in $\langle x_i, x_{i+1} \rangle$.

Then there is an appropriate embedding of G: Placing x in place of z and y sufficiently close to x maintains convexity. (No three vertices are collinear; $|N(x)|, |N(y)| \geq 3$.)

Case 2b. For every i there is a vertex in N_y that is not contained in $\langle x_i, x_{i+1} \rangle$.

Then there must be a $v \in N_y$ that is not on $\langle x_i, x_{i+1} \rangle$ and x, y, x_i, x_{i+1}, u, v are the branch vertices of a $K_{3,3}$-subdivision in G.