Upper bounds

Theorem. $\chi(G) \leq \Delta(G) + 1$.

Proof. Algorithmic. Greedy coloring.

Definition A graph is called r-degenerate, if there is an ordering $v_1, v_2, \ldots, v_n$ of the vertices such that for every $i = 1, 2, \ldots, n-1$, we have

$$|N(v_{i+1}) \cap |\{v_1, \ldots, v_i\}| \leq r.$$

Theorem. If G is r-degenerate, then $\chi(G) \leq r + 1$.

Proof. Greedy coloring using the special vertex order.

Corollary Interval graphs are perfect

Proof. Interval graphs are $(\omega(G) - 1)$-degenerate.

Jordan Curves

For continuous $\gamma : [0,1] \to I\!R^2$, the subset

$$Im(\gamma) := \{\gamma(x) : x \in [0,1]\} \subseteq I\!R^2$$

is called a curve. $\gamma(0)$ and $\gamma(1)$ are called the *end-points* of the curve.

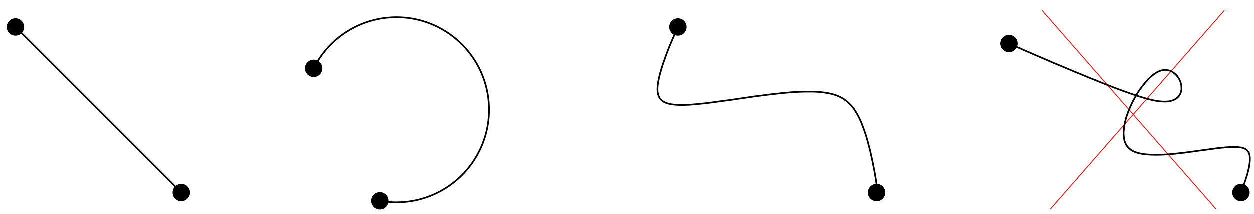
A curve is closed if $\gamma(0) = \gamma(1)$.
A curve is simple if it is injective (except possibly $\gamma(0) = \gamma(1)$).

A closed simple curve is called a Jordan-curve.

Examples: Line segments between $p, q \in I\!R^2$

$$x \mapsto xp + (1-x)q \ ,$$

circular arcs, Bezier-curves without self-intersection are simple curves*



*The Peano curve is not simple.

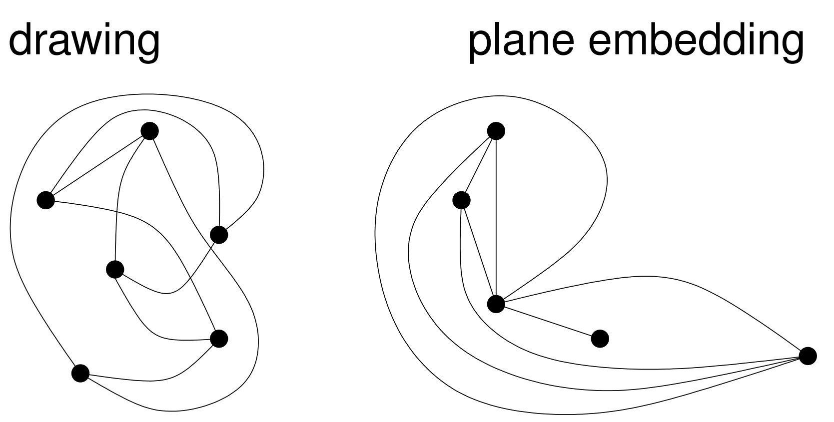
Drawing of graphs

A drawing of a multigraph G is a function f defined on $V(G) \cup E(G)$ such that

- $f|_{V(G)} : V(G) \to \mathbb{R}^2$ is injective and
- $f(uv)$ is an $f(u), f(v)$-curve for every $uv \in E(G)$.

A point in $f(e) \cap f(e')$ that is not a common endpoint of e and e' is called a crossing.

A multigraph is planar if it has a drawing without crossings. Such a drawing is a planar embedding of G. A planar (multi)graph *together* with a particular planar embedding is called a plane (multi)graph.



Are there non-planar graphs?

Proposition. K_5 and $K_{3,3}$ cannot be drawn without crossing.

Proof. Define the *conflict graph* of edges.

The unconscious ingredient.

Jordan Curve Theorem. If C is a Jordan curve, then $\mathbb{R}^2 \setminus C$ is the disjoint union of two nonempty path-connected* open† sets‡. One of these is bounded (the *interior* of C), the other is unbounded (the *exterior* of C) and both have C as their boundary.

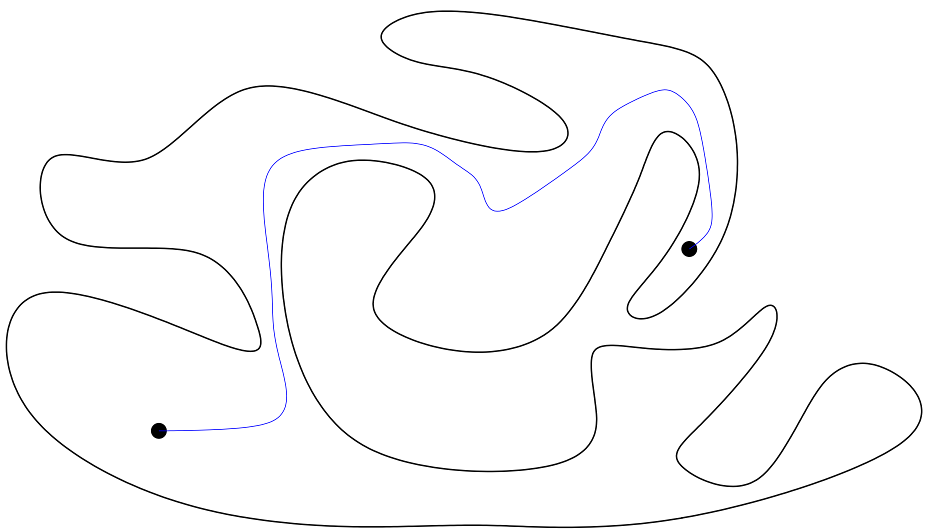
Remark Every curve between a point of the exterior and a point of the interior contains a point of C.

$^*U \subseteq \mathbb{R}^2$ is path-connected if for $\forall u, v \in U \;\; \exists$ a u, v-curve in U

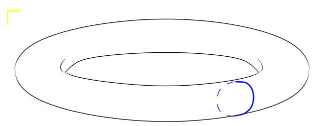
$^\dagger U \subseteq R^2$ is an open if for every $p \in U$ there is an $\epsilon > 0$ such that the disk of radius ϵ with center p is contained in U.

‡A path-connected open set is called a region.

Jordan Curve Theorem



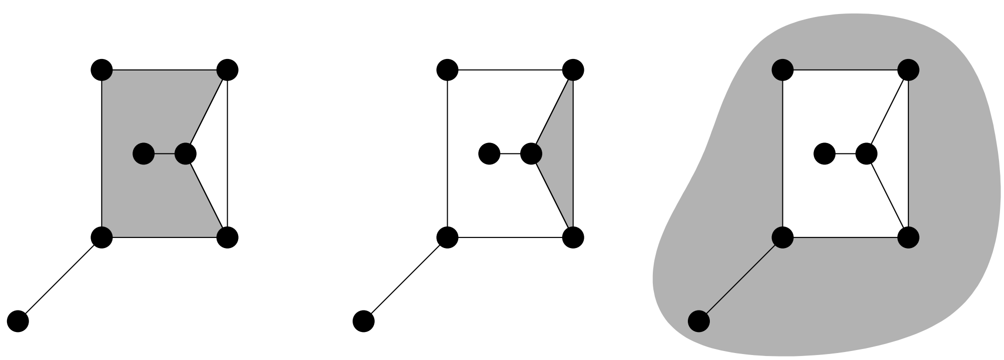
JCT is not true on the torus!



Faces

The faces of a plane multigraph are the maximal regions of the plane that contain no points used in the embedding.

A finite plane multigraph G has one unbounded face (also called outer face).



Proposition If G is a connected plane multigraph with $\delta(G) \geq 2$ then the boundary of each face forms a cycle in G. If $l(F_i)$ denotes the length of face F_i then

$$2e(G) = \sum l(F_i).$$

Euler's Formula

Theorem.(Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Proof. Induction on e.

Base Case. If $e = 0$, then $n = k$ and $f = 1$.

Suppose now $e > 0$.

Case 1. G has a cycle.

Delete one edge from a cycle. In the new graph:

$e' = e - 1$, $n' = n$, $f' = f - 1$ (Jordan!), and $k' = k$.

Case 2. G is a forest.

Delete a pendant edge. In the new graph:

$e' = e - 1$, $n' = n$, $f' = f$, and $k' = k + 1$.

Application – Platonic solids

- each face is congruent to the same regular convex r-gon, $r \geq 3$
- the same number d of faces meet at each vertex, $d \geq 3$

EXAMPLES: cube, tetrahedron

$$fr = 2e \qquad vd = 2e$$

Substitute into Euler's Formula

$$\frac{2e}{d} - e + \frac{2e}{r} = 2$$

$$\frac{1}{d} + \frac{1}{r} = \frac{1}{2} + \frac{1}{e}$$

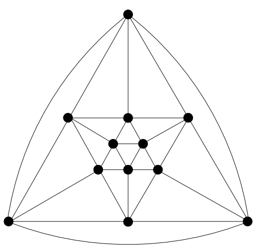
Crucial observation: either d or r is 3.

Possibilities: $r \quad d \quad e \quad f \quad v$

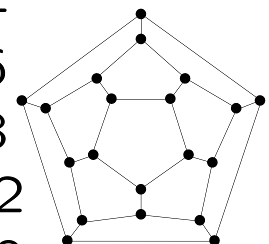
Applications of Euler's Formula

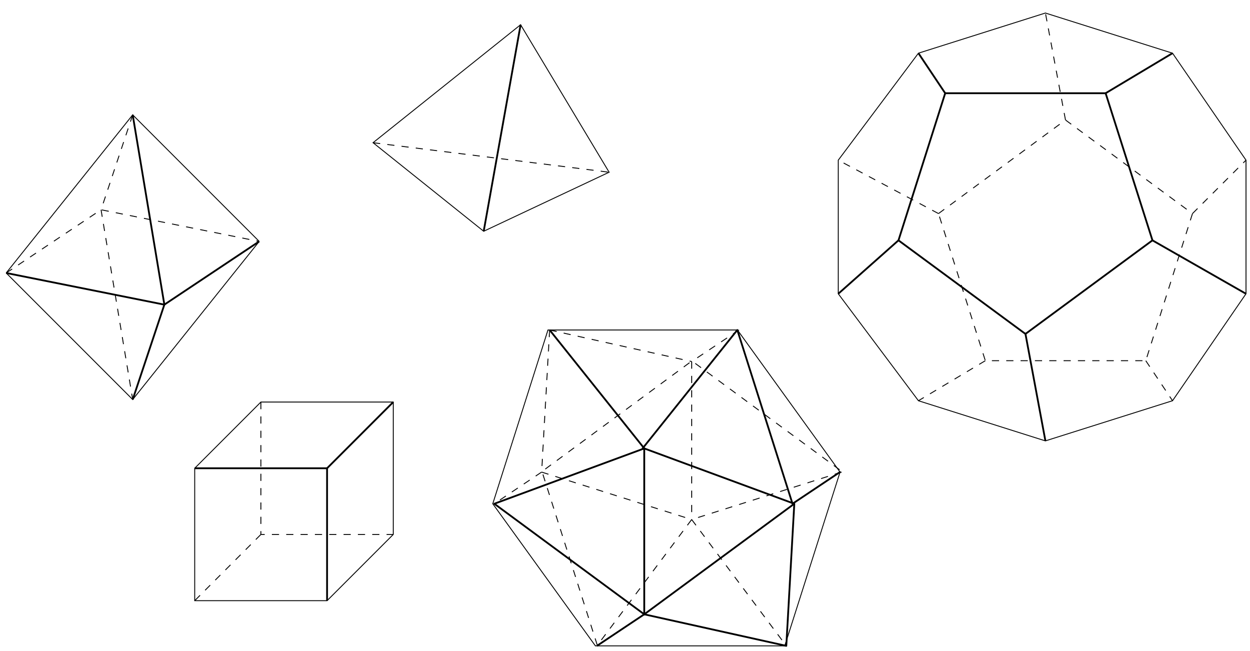
For a convex polytope,

$$\#\text{Vertices} - \#\text{Edges} + \#\text{Faces} = 2$$



Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20





The platonic solids

When is a graph planar?

Corollary If G is a simple, planar graph with $v(G) \geq 3$, then $e(G) \leq 3v(G) - 6$.
If also G is triangle-free, then $e(G) \leq 2v(G) - 4$.

Corollary K_5 and $K_{3,3}$ are non-planar.

The subdivision of edge $e = xy$ is the replacement of e with a new vertex z and two new edges xz and zy. The graph H' is a subdivision of H, if one can obtain H' from H by a series of edge subdivisions. Vertices of H' with degree at least three are called branch vertices.

Theorem (Kuratowski, 1930) A graph G is planar iff G does not contain a subdivision of K_5 or $K_{3,3}$.

Dual graph

Denote the set of faces of a plane multigraph G by $F(G)$ and let $E(G) = \{e_1, \ldots, e_m\}$. Define the dual multigraph G^* of G by

- $V(G^*) := F(G)$
- $E(G^*) := \{e_1^*, \ldots, e_m^*\}$, where the endpoints of e_i^* are the two (not necessarily distinct) faces $f', f'' \in F(G)$ on the two sides of e_i.

Remarks. Multiple edges and/or loops *could* appear in the dual of simple graphs

Different planar embeddings of the *same* planar graph could produce *different* duals.

A maximal planar graph is a simple planar graph that is not a spanning subgraph of another planar graph. A triangulation is a simple plane graph where every face is a triangle.

Proposition. For a simple n-vertex plane graph G, the following are equivalent.

$A)$ G has $3n - 6$ edges

$B)$ G is a triangulation.

$C)$ G is a maximal planar graph.