Pigeonhole Principle

Q: Are there two people in Berlin with the SAME number of strands of hair? ⤳ YES

Informal PHP (Schubfach Prinzip)
When $n+1$ pigeons sit in n pigeonholes then $\exists$ a pigeonhol with at least 2 pigeons.

Formal PHP X, Y finite sets with $|X| > |Y|$

$\forall f: X \to Y \quad \exists x_1, x_2 \in X, x_1 \neq x_2$ s.t. $f(x_1) = f(x_2)$

(X ↳ pigeons, Y ↳ pigeonholes)

Examples/Applications

(1.) Among 13 people there are two who were born in the same month.

(2.) $B \subseteq [200]$, $|B| = 101 \Rightarrow \exists x \neq y \in B$ s.t. $x \mid y$

Remark: best possible, as it is false with $|B| = 100$, e.g when $B = \{101, 102, \dots, 200\}$.

Proof What should be the pigeonholes?

$x \in [200] \rightsquigarrow x = 2^{s_x} \cdot b_x$ with $s_x \in \mathbb{N}_0$, b_x odd.

Put $\underset{\text{pigeons}}{X = B}$, $\underset{\text{pigeonholes}}{Y = \text{odd numbers in } [200]}$

$$f: X \to Y, \quad x \mapsto b_x \qquad |X| > |Y| \Rightarrow \exists x_1 \neq x_2 \in B \text{ with } b_{x_1} = b_{x_2}$$

Say $x_1 < x_2 \Rightarrow s_{x_1} < s_{x_2} \Rightarrow x_1 = 2^{s_{x_1}} \cdot b_{x_1} \mid 2^{s_{x_2}} \cdot b_{x_2} = x_2$

as $\frac{x_2}{x_1} = 2^{s_{x_2} - s_{x_1}}$ ∎

(3.) **Thm**

Chinese Remainder Theorem

$\forall m, n \in \mathbb{N} \quad \gcd(m, n) = 1$

$\forall a, b \in \mathbb{N}_0 \quad 0 \leq a < m, \; 0 \leq b < n$

$\exists\, x \in \mathbb{Z}$ s.t. $x \equiv a \pmod m$ and $x \equiv b \pmod n$
$(\exists\, q \in \mathbb{Z} : x = q \cdot m + a)$ $(\exists\, r \in \mathbb{Z} : x = r \cdot n + b)$

Example

$\exists ?\, x \in \mathbb{N}$ s.t. $x \equiv 26 \pmod{63}$, $x \equiv 13 \pmod{40}$

$\Longrightarrow$ YES! (since $\gcd(63,40) = 1$) e.g. $x = 1853$

Proof (CHRT)

$A = \{a, m+a, 2m+a, \dots, (n-1)\cdot m + a\}$

Then $|A| = n$ and every element of A is $\equiv a \pmod m$

Case 1: $\exists\, x \in A$ s.t. $x \equiv b \pmod n$ ✓

Case 2: $\forall\, x \in A$ $x \not\equiv b \pmod n$

$\underset{\text{pigeons}}{X = A}$, $\underset{\text{pigeonholes}}{Y = \{0, 1, \dots, n-1\} \setminus \{b\}}$
$f : X \longrightarrow Y$
$x \longmapsto$ remainder $\pmod n$

$\Longrightarrow \exists\, x_1, x_2 \in A$, $x_1 \neq x_2$, $x_1 \equiv x_2 \pmod n$

$\exists\, i_1 < i_2$, $i_1, i_2 \in \{0, 1, \dots, n-1\}$ s.t. $x_1 = i_1 \cdot m + a$, $x_2 = i_2 \cdot m + a$

$\Longrightarrow i_1 \cdot m + a \equiv i_2 \cdot m + a \pmod n \Longrightarrow (i_2 - i_1)\, m = 0 \pmod n$

$\Longrightarrow n \mid (i_2 - i_1)\cdot m \Longrightarrow$ as $\gcd(m,n) = 1$: $n \mid i_2 - i_1$

However $0 \leq i_2 - i_1 < n \rightsquigarrow$ we must have $i_1 = i_2$ ↯ ∎

Pigeonhole Principle - General form

Let $q_1, \dots, q_n \in \mathbb{N}$. When $1 + \sum_{i=1}^{n} (q_i - 1)$ objects are placed in n (numbered) boxes, then $\exists\, i \in [n]$ s.t. Box i contains at least q_i objects.

Proof - By contradiction: If the statement were not true, then $\forall i \in [n]$ Box i would contain less then q_i objects $\Rightarrow$ together we could place at most $\sum_{i=1}^{n}(q_i - 1)$ objects ↯. ■

Remark:
When applied, either write out the setting in detail or provide the proof.

Special case $q_1 = \cdots = q_n = q \in \mathbb{N}$
- when n boxes together hold $n(q-1)$ objects then $\nexists$ a Box with q objects.
- Reformulation (Averaging): when n boxes together hold Q objects, then
  - $\exists$ Box with at least $\lceil \frac{Q}{n} \rceil$ objects
  - $\exists$ Box with at most $\lfloor \frac{Q}{n} \rfloor$ objects

! PHP is only an EXISTENCE proof technique. The algorithmic problem is much harder, no general method known.

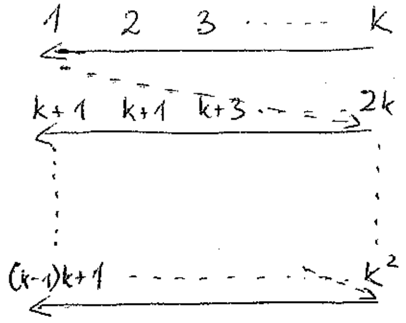
Qestion Given a sequence of length n, how long monotone subsequence can we find in it? (can we guarantee?)

E.g.

n	2	3	4	5	6	7	8	9
	2	2	2	3	3	3	3	3

- we can assume that all elements are distinct (equality just "helps")

Construction for $n = k^2$:



Read rows backwards one after the other.

Claim: Longest monotone subsequence is of length k.

Proof → at least k ✓

→ at most k: ∘ every increasing subsequence can have at most one element from each row ⇝ length is at most k.

∘ every decreasing subsequence can have at most one element from each column ⇝ length is at most k. ▨

Now we prove that this construction is best possible:

Thm (Erdős - Szekeres, 1930's)

Every sequence of k^2+1 reals contains a monotone subsequence of length $k+1$.

Proof $x_1, x_2, \ldots, x_{k^2+1}$

For x_j let i_j be the length of the longest increasing subsequence ending at x_j.

<u>Case 1</u> $\exists x_j$ s.t. $i_j \geq k+1 \rightsquigarrow \checkmark$

<u>Case 2</u> $\forall x_j$ we have $i_j \leq k$.

Objects (pigeons): $x_1, \dots, x_{k^2+1}$

Boxes (pigeonholes): $I_p = \{x_j : i_j = p\}$ $1 \leq p \leq k$

By averaging $\exists$ box I_r s.t.

$|I_r| \geq \left\lceil \frac{k^2+1}{k} \right\rceil = k+1$

Claim: Elements of I_r form a decreasing sequence of length $k+1$.

Indeed, $\forall x_\ell, x_j \in I_r$, $\ell < j$ we must have $x_\ell \geq x_j$, otherwise $r = i_\ell < i_j = r$ ↯. ■

Further (easy) application of the PHP:

$\forall$ coloring of $2k-1$ points with 2 colors contains k points with the same color. (this is also best possible)

In many applications there are two element relations, so the question also makes sense if we color 2-element subsets.

<u>Q:</u> How many points do we need to have to guarantee k points such that all pairs from them are of the same color?

<u>Example</u> $k=3 \rightsquigarrow$ 5 not enough, 6 is enough:

<u>Def</u> Let V be a set. Given a Red-Blue coloring of $\binom{V}{2}$, a subset $K \subseteq V$ is called

monochromatic (m.c.) if every set in $\binom{k}{2}$ has the same color.

<u>Prop</u> Given a Red-Blue coloring of $\binom{[6]}{2}$ $\exists$ m.c. $M \subseteq [6]$, $|M| = 3$.

<u>Proof</u> Look at 12, 13, 14, 15, 16.

By the PP (averaging): $\exists \lceil \frac{5}{2} \rceil = 3$ of the same color.

$2 \leq i < j < k \leq 6$ $1i, 1j, 1k$ are all say Red.

Now look at ij, ik, jk.

Case 1: $\exists$ pair among them of color Red.

↳ say $ij \Rightarrow M = \{1, i, j\}$ is m.c.

Case 2: $\forall$ pair among them is of color Blue.

$\rightsquigarrow M = \{i, j, k\}$ is m.c. ▨

How about $k = 4$? 18 enough, 17 not (much harder)

In general:

<u>Def</u> Ramsey number, $k \geq 2$

$R(k) = \min \{ N : \forall c : \binom{[N]}{2} \longrightarrow \{R, B\} \ \exists \text{ m.c } K \subseteq [N], |K| = k \}$

<u>Example</u>

$R(2) = 2$, $R(3) = 6$

<u>Thm</u> (Ramsey) $R(k) < 4^k$

In particular: $R(k) < \infty$