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PRACTICE EXAM - solution

Exercise 1 [10 points]

(1) State the definition of the Ramsey number $R(k)$.

(2) Prove that $R(4) \leq 32$.

Solution (1)

$$R(k) = \min\left\{N \ : \ \forall c: \binom{[N]}{2} \to \{\text{Red,Blue}\} \ \exists K \subseteq \binom{[N]}{k}, \ K \text{ is monochromatic with respect to } c\right\},$$

where a subset S of $[N]$ is monochromatic w.r.t. c if all elements of $\binom{S}{2}$ have the same c-colour.

(2) Let c be an arbitrary Red/Blue colouring of the edges of the complete graph on 32 vertices. Let $V_1 = [32]$ be the vertex set of the graph, and let $i_1 = 1$. Define

$$R_1 = \{j \in V_1 \ : \ c(i_1 j) = \text{Red}\} \ \text{ and } \ B_1 = \{j \in V_1 \ : \ c(i_1 j) = \text{Blue}\}.$$

By averaging, either R_1 or B_1 has size at least $\lceil \frac{32-1}{2} \rceil = 16$. Let $V_2 \subseteq V_1$ be this larger set and C_1 the corresponding color. Then $|V_2| \geq 16$ and for every $j \in V_2$ we have $c(i_1 j) = C_1$.

Next let i_2 be an arbitrary vertex in V_2 (say $\min V_2$),

$$R_2 = \{j \in V_2 \ : \ c(i_2 j) = \text{Red}\} \ \text{ and } \ B_2 = \{j \in V_2 \ : \ c(i_2 j) = \text{Blue}\}.$$

By averaging either R_2 or B_2 has size at least $\lceil \frac{16-1}{2} \rceil = 8$. Let $V_3 \subseteq V_2$ be this larger set and C_2 the corresponding color. Then $|V_3| \geq 8$ and for every $j \in V_3$ we have $c(i_2 j) = C_2$.

Next let $i_3 = \min V_3$,

$$R_3 = \{j \in V_3 \ : \ c(i_3 j) = \text{Red}\} \ \text{ and } \ B_3 = \{j \in V_3 \ : \ c(i_3 j) = \text{Blue}\}.$$

By averaging either R_3 or B_3 has size at least $\lceil \frac{8-1}{2} \rceil = 4$. Let $V_4 \subseteq V_3$ be this larger set and C_3 the corresponding color. Then $|V_4| \geq 4$ and for every $j \in V_4$ we have $c(i_3 j) = C_3$.

Next let $i_4 = \min V_4$,

$$R_4 = \{j \in V_4 \; : \; c(i_4 j) = \text{Red}\} \;\text{ and }\; B_4 = \{j \in V_4 \; : \; c(i_4 j) = \text{Blue}\}.$$

By averaging either R_4 or B_4 has size at least $\lceil \frac{4-1}{2} \rceil = 2$. Let $V_5 \subseteq V_4$ be this larger set and C_4 the corresponding color. Then $|V_5| \geq 2$ and for every $j \in V_5$ we have $c(i_4 j) = C_4$.
Next let $i_5 = \min V_5$,

$$R_5 = \{j \in V_5 \; : \; c(i_5 j) = \text{Red}\} \;\text{ and }\; B_5 = \{j \in V_5 \; : \; c(i_5 j) = \text{Blue}\}.$$

By averaging either R_5 or B_5 has size at least $\lceil \frac{2-1}{2} \rceil = 1$. Let $V_6 \subseteq V_5$ be this larger set and C_5 the corresponding color. Then $|V_6| \geq 1$ and for every $j \in V_6$ we have $c(i_5 j) = C_5$.
To finish let $i_6 = \min V_6$.
Now consider $C_1, C_2, C_3, C_4, C_5 \in \{\text{Red,Blue}\}$. By averaging either Red or Blue appears at least $\lceil 5/2 \rceil = 3$ times. W.l.o.g assume that it is Red, and let $C_{j_1} = C_{j_2} = C_{j_3} = \text{Red}$, with $j_1 < j_2 < j_3$. We claim that $\{i_{j_1}, i_{j_2}, i_{j_3}, i_{j_4} = i_6\}$ is a Red monochromatic 4-set. Indeed, let $\ell, \ell' \in \{j_1, j_2, j_3, j_4\}$ with $\ell < \ell'$. Then as $i_{\ell'} \in V_{\ell'} \subseteq V_{\ell+1}$ we have $c(i_\ell i_{\ell'}) = C_\ell = \text{Red}$.

Exercise 2 [10 points]
Prove that among a group of n people, for any integer $n \geq 2$, there will always be two people who know exactly the same number of people (assuming that knowing is a mutual relationship).

Solution *First solution* Note that the number of people one person can know can take n different values, namely $0, 1, 2, \ldots, n-1$. Now assume that there are no two people who know exactly the same number of people. This is possible only if every possible value from before appears exactly once. However then there is some person A, who knows everyone, and in particular he also knows some person B, who knows exactly 0 people, which is a contradiction.
Second solution We divide our problem into two cases. *Case 1*: Everyone knows at least one person. Then the number of persons that a person knows can be any number from $1, 2, \ldots, n-1$. Since there are n people (pigeons), by the pigeonhole principle, at least two people know the same number of people. *Case 2*: There exists a person who does not know anyone. Then no person can know everyone, and hence the number of persons that a person knows can be any number from $0, 2, \ldots, n-2$. Again by the pigeonhole principle we get two people who know the same number of people.

Exercise 3 [10 points]
Let G be a graph which is k-regular, i.e., every vertex has degree k. Prove that if there are no cycles of length 3 and 4 in G[1], then G must have at least k^2+1 vertices.

Solution Let $x \in V(G)$ be an arbitrary vertex and let $x_1, x_2 \ldots, x_k \in V(G)$ be its k neighbors. Now for $1 \leq i \leq k$, let $x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}$ be the neighbors of x_i different

[1] The Petersen graph that you saw in the lectures is an example of such a graph with $k = 3$.

from x. We claim that all these $k+1+k(k-1)=k^2+1$ vertices are different and hence $|V(G)| \geq k^2+1$. By construction
– $x \neq x_i$ for $1 \leq i \leq k$,
– $x \neq x_{i,j}$ for $1 \leq i \leq k$ and $1 \leq j \leq k-1$,
– $x_i \neq x'_i$ for $1 \leq i \neq i' \leq k$,
– $x_i \neq x_{i,j}$ for $1 \leq i \leq k$ and $1 \leq j \leq k-1$,
– $x_{i,j} \neq x_{i,j'}$ for $1 \leq i \leq k$ and $1 \leq j \neq j' \leq k-1$.
Hence the only remaining case to exlude are:
$x_i = x_{i',j}$ for some $1 \leq i \neq i' \leq k$, $1 \leq j \leq k-1$: Then $x, x_i = x_{i',j}, x'_i$ is a cycle of length 3, which is a contradiction.
$x_{i,j} = x_{i',j'}$ for some $1 \leq i \neq i' \leq k$, $1 \leq j, j' \leq k-1$: Then $x, x_i, x_{i,j} = x_{i',j'}, x'_i$ is a cycle of length 4, which is a contradiction.

Exercise 4 [10 points]

(1) Define a poset and an antichain in a poset.

(2) Prove that in the boolean poset $(2^{[n]}, \subseteq)$ the largest antichain has size $\binom{n}{\lfloor n/2 \rfloor}$.

Solution (1) A poset is a set P together with a relation $\leq$ which is
– reflexive ($a \leq a$ for all $a \in P$),
– antisymmetric (if for $a, b \in P$ we have $a \leq b$ and $b \leq a$ then $a = b$),
– transitive (if for $a, b, c \in P$ we have $a \leq b$ and $b \leq c$ then $a \leq c$).
(2) An antichain of this size clearly exists as we can take all subsets of size $\lfloor \frac{n}{2} \rfloor$. To prove that this is largest possible take an arbitrary antichain $\mathcal{F}$ and double count the total number M of those pairs (π, F) where π is a permutations of $[n]$ and $F \in \mathcal{F}$ is an initial segment of π, i.e., $\{\pi(1), \ldots, \pi(|F|)\} = F$.
On one hand, for every $F \in \mathcal{F}$ there are $|F|!(n-|F|)!$ permutations $\pi \in S_n$ with $\{\pi(1), \ldots, \pi(|F|)\} = F$, and so

$$M = \sum_{F \in \mathcal{F}} |F|!(n-|F|)!.$$

On the other hand, for every $\pi \in S_n$, as $\mathcal{F}$ is an antichain, there is at most one k such that $\{\pi(1), \ldots, \pi(k)\} \in \mathcal{F}$, so

$$M \leq n!.$$

Hence

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)! \leq n!$$

and so

$$1 \geq \sum_{F \in \mathcal{F}} \frac{|F|!(n-|F|)!}{n!} = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \geq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = |\mathcal{F}| \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \Rightarrow |F| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

where we used that for every $F \in \mathcal{F}$ we have $\binom{n}{|F|} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Exercise 5 [10 points]

(1) State the definition of the Striling number of second kind.

(2) Let $S(n,r)$ be the Stirling number of the second kind. Prove that

$$r!S(n,r) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} i^n.$$

Solution (1) $S(n,r)$ is the number of r-partitions of $[n]$, i.e.

$$S(n,r) = \left| \left\{ \{X_1, \ldots, X_r\} \ : \ \begin{array}{c} \emptyset \neq X_i \subseteq [n] \text{ for } 1 \leq i \leq r, \\ X_i \cap X_j = \emptyset \text{ for } 1 \leq i \neq j \leq r \text{ and} \\ X_1 \cup \cdots \cup X_r = [n] \end{array} \right\} \right|.$$

(2) Let M be the set of all surjective functions $f : [n] \to [r]$. We will do a double counting for M.
The collection of the preimages of the elements $i \in [r]$ form a r-partition of $[n]$, and if we chose some r-partition, then there are $r!$ ways to assign the values from $[r]$ to them. Hence $|M| = r!S(n,r)$.
Now let S be the set of all $[n] \to [r]$ functions; then $|S| = r^n$ since each element of $[n]$ has r possible independent images. For $i \in [r]$ let E_i be the collection of those functions which do not take the value i; then $|E_i| = r^{n-1}$ for all i. Then $M = S \setminus \left(\cup_{i\in[r]} E_i\right)$. By the inclusion-exclusion principle

$$|M| = \left| S \setminus \left(\cup_{i\in[r]} E_i\right) \right| = \sum_{I \subseteq [r]} (-1)^{|I|} \left| \cap_{i \in I} E_i \right|.$$

However for $I \subseteq [r]$ the set $\cap_{i\in I} E_i$ is the collection of all $[n] \to [r] \setminus I$ functions, so $\left| \cap_{i\in I} E_i \right| = (r - |I|)^n$ since every element of $[n]$ has $(r - |I|)$ possible images. Therefore

$$|M| = \sum_{I\subseteq[r]} (-1)^{|I|} (r - |I|)^n = \sum_{i=0}^{r} \sum_{I \subseteq [n], |I|=i} (-1)^i (r-i)^n = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (r-i)^n = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} i^n,$$

where in the last step we have used $\binom{r}{i} = \binom{r}{r-i}$ and the substitution $i \mapsto r - i$.

Exercise 6 [10 points]
For $n \in \mathbb{N}$, let p_n denote the number of permutations of $[n]$ whose cycle decomposition only contains cycles of length at most 2.

(1) Find a recurrence relation for the sequence $(p_n)_{n\geq 0}$.

(2) By solving this recurrence, obtain an explicit formula for p_n.

Solution (1) Classify the permutations according to the length of the cycle n is in. If n is a fixed point, then the permutation restricted to the rest is a permutation of $[n-1]$ whose cycle decomposition only contains cycles of length at most 2, and hence their number is p_{n-1}. If n is in a cycle of length 2, then there are $n-1$ ways to chose the other element i in this cycle and for given i the permutation restricted to $[n-1] \setminus \{i\}$ is a permutation of $[n-1] \setminus \{i\}$ whose cycle decomposition only contains cycles of length at most 2, and hence their number is p_{n-2}. Since these two cases form a partition of the set that we need to count, for $n \geq 2$ we get

$$p_n = p_{n-1} + (n-1)p_{n-2} \qquad (p_0 = p_1 = 1).$$

(2) Let $\hat{P}(x)$ be the exponential generating function of p_n. By the recursion, we get

$$\sum_{n=0}^{\infty} p_{n+2} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} p_{n+1} \frac{x^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty} (n+1)p_n \frac{x^{n+1}}{(n+1)!},$$

which is equivalent to

$$\begin{aligned} \sum_{n=0}^{\infty} \left(p_{n+2} \frac{x^{n+2}}{(n+2)!} \right)' &= \sum_{m=1}^{\infty} p_m \frac{x^m}{m!} + x \sum_{n=0}^{\infty} p_n \frac{x^n}{(n)!} \\ \left(\hat{P}(x) - p_0 - p_1 x \right)' &= \hat{P}(x) - 1 + x\hat{P}(x) \\ \hat{P}'(x) - p_1 &= \hat{P}(x) - 1 + x\hat{P}(x) \end{aligned}$$

Now substituting $p_1 = 1$, and simplifying, we get

$$\begin{aligned} \left(\ln \hat{P}(x) \right)' = \frac{\hat{P}'(x)}{\hat{P}(x)} &= 1 + x \\ \ln \hat{P}(x) &= x + \frac{x^2}{2} + c \\ \hat{P}(x) &= e^c e^{x + \frac{x^2}{2}}. \end{aligned}$$

As $\hat{P}(0) = p_0 = 1$ we have $c = 0$, so

$$\hat{P}(x) = e^{x+\frac{x^2}{2}} = e^x e^{\frac{x^2}{2}} = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} \frac{1}{2^\ell} \frac{x^{2\ell}}{\ell!} \right) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{k!\ell!2^\ell} x^{k+2\ell}.$$

Taking $n = k + 2\ell$, the expression is equal to

$$\sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^\ell} \frac{1}{\ell!} \frac{1}{(n-2\ell)!} \right) x^n.$$

Hence

$$p_n = \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^\ell \ell! (n-2\ell)!}.$$