Van der Waerden's Theorem

An r-coloring of a set S is a function $c : S \to [r]$.
A set $X \subseteq S$ is called monochromatic if c is constant on X.

Let $\mathbb{N}$ be two-colored.
Is there a monochromatic 3-AP?

Roth's Theorem says: YES, in the larger of the two color classes.

A weaker statement, not specifying in which color the 3-AP occurs:

Proposition In every two-coloring of $\left[2 \cdot (5 \cdot (2^5 + 1))\right]$ there is a monochromatric 3-AP.

What if we want a longer arithmetic progression?
Can we color the integers with two colors such that there is no monochromatic 4-AP?
Szemerédi's Theorem says NO.

How far must we color the integers to find an AP of length 4? Or k?

In order to prove something about this, we introduce more colors.

$W(r, k)$ is the smallest integer w such that any r-coloring of $[w]$ contains a monochromatic k-AP.

Theorem (van der Waerden, 1927) For every $k, r \geq 1$, $W(r, k) < \infty$.

Remark Consequence of Szemerédi's Theorem.

Proof of Van der Waerden's Theorem

Induction on k, the following statement:
"For all $r \geq 1$, $W(r,k) < \infty$"

$W(r,1) = 1$
$W(r,2) = r+1$
$W(r,3) = ?$

Suppose $W(r,k) < \infty$ for every $r \geq 1$.
Let us find an upper bound on $W(r,k+1)$ in terms of these numbers.

$$W(1,k+1) = k+1$$

$$W(2,k+1) \leq 2 \cdot (2W(2,k)) \cdot W(2^{2W(2,k)}, k)$$

$$W(3,k+1) \leq 2 \cdot 2 \cdot 2W(3,k) \cdot W(3^{2W(3,k)}, k) \cdot W(3^{2\cdot(2W(3,k))\cdot W(3^{2W(3,k)},k)}, k)$$

For general r, define the (kind of fast growing) function $L_r : \mathbb{N} \to \mathbb{N}$,

$$L_r(x) = xW(r^x, k).$$

Then

$$W(r, k+1) \leq \underbrace{2L_r(\cdots 2L_r(2L_r(2L_r(1))))}_{r\text{-times}}.$$

We prove by induction on i, that no matter how the first $x_i = \underbrace{2L_r(\cdots 2L_r(2L_r(2L_r(1))))}_{i\text{-times}}$ integers are colored with r colors, there exists i monochromatic k-APs $a^{(j)}, a^{(j)} + d_j, \ldots, a^{(j)} + (k-1)d_j$, $1 \leq j \leq i$, each in different colors, such that $a^{(j)} + kd_j$ is the very same integer a for each j, $1 \leq j \leq i$.

Divide $[L_r(x_i)]$ into blocks of x_i integers. There are r^{x_i} ways to r-color a block. By the definition of $W(r^{x_i}, k)$, there is a k-AP of blocks with the same coloring pattern.

Let c_j be the color of the monochromatic k-AP $a^{(j)}, a^{(j)} + d_j, \ldots, a^{(j)} + (k-1)d_j$, for $1 \leq j \leq i$.

Case 1. If the color of $a = a^{(j)} + kd_j$ is one of these colors then there is a $(k+1)$-AP in this color and we are done.

Case 2. Otherwise the copies of a in the k blocks forms a monochromatic k-AP of color $c_{i+1} \neq c_j$, $1 \leq j \leq i$. We can form monochromatic k-APs in the other colors c_j: Take the copy of $a^{(j)} + (l-1)d_j$ from the l^{th} block.

These $i+1$ k-APs are monochromatic of $i+1$ distinct colors and would be continued in the same $(k+1)^{st}$ element. This element is certainly less than $2L_r(x_i)$.

After the rth iteration the colors run out, Case 2 cannot occur, and we have a monochromatic $(k+1)$-AP.
□

Turán-type questions

We are looking for a subtructure of a given size.

Turán-type problems: How large fraction of the structure will surely contain a given substructure?

Most natural special case: we are looking for a smaller "copy" of the structure itself.

Turán's Theorem
Structure: $E(K_n)$
Substructure: $E(K_k)$
Statement:
$F \subseteq E(K_n)$, $|F| \geq \left(1 - \frac{1}{k-1}\right)\binom{n}{2} \Rightarrow F \supseteq E(K_k)$

Szemerédi's Theorem
Structure: $[n]$
Substructure: k-AP
Statement: $S \subseteq [n]$, $|S| \geq \frac{n}{f(n)} \Rightarrow S$ contains a k-AP
(for some function $f : \mathbb{N} \to \mathbb{N}$, $f(n) \to \infty$.)

Ramsey-type questions

Ramsey-type problems: How large should the structure be such that in any given r-coloring there is a given substructure that is monochromatic?

Van der Waerden's Theorem (Counterpart of Szemerédi's Theorem)
Structure: $[n]$
Substructure: k-AP
Statement: If n is large enough, then there is a monochromatic k-AP in any r-coloring of $[n]$

Ramsey's Theorem (Counterpart of Turán's Theorem)
Structure: $E(K_n)$
Substructure: $E(K_k)$
Statement: If n is large enough, then there is a monochromatic $E(K_k)$ in any r-coloring of $E(K_n)$

Ramsey's Theorem

Proposition In any RED/BLUE-coloring of $E(K_6)$ there is either a RED K_3 or a BLUE K_3

Let $R(k)$ be the smallest integer R such that any two-coloring of $E(K_R)$ contains a monochromatic copy of K_k.

Proposition $R(3) = 6$

To prove the existence of $R(k)$ in general we need a bit more general notion.

Let $R(k, l)$ be the smallest integer R such that any RED/BLUE-coloring of $E(K_R)$ contains a RED copy of K_k or a BLUE copy of K_l.

Ramsey's Theorem For any $k, l \geq 1$, we have

$$R(k, l) < \infty.$$

Proof of Ramsey's Theorem

Induction on $k + l$.
$R(k, 1) = R(1, l) = 1$
$R(k, 2) = k$, $R(2, l) = l$

Claim $R(k, l) \leq R(k - 1, l) + R(k, l - 1)$

Proof. Let $R = R(k - 1, l) + R(k, l - 1)$ and let c be an arbitrary RED/BLUE-coloring of $E(K_R)$.

Let $x \in V(K_R)$ arbitrary.
Define $N_{blue} = \{v \in V(K_R) : xv \text{ is BLUE}\}$ and
$N_{red} = \{v \in V(K_R) : xv \text{ is RED}\}$

Since $|N_{blue}| + |N_{red}| + 1 = R$ we have either $|N_{blue}| \geq R(k, l - 1)$ or $|N_{red}| \geq R(k - 1, l)$.

If $|N_{blue}| \geq R(k, l - 1)$ then there is a RED K_k or a BLUE K_{l-1} which together with x forms a BLUE K_l.

If $|N_{red}| \geq R(k - 1, l)$ then there is a BLUE K_l or a RED K_{k-1} which together with x forms a RED K_k. □

Corollaries

Corollary $R(k,l) \leq \binom{k+l-2}{k-1}$

Proof. By induction on $k+l$.

$$\begin{aligned} R(k,l) &\leq R(k-1,l) + R(k,l-1) \\ &\leq \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1} = \binom{k+l-2}{k-1} \end{aligned}$$

Corollary $R(k) < 4^k$

$1000 dollar question: How large is $R(k)$?

Lower bounds

Lower bound The "Turán-graph coloring" gives $(k-1)^2 < R(k)$.

"Construction" (Erdős, 1951) $\sqrt{2}^k < R(k)$

Strategy: Count **all** two-colorings of $E(K_n)$ as well as the "**bad**" ones (the ones that contain a monochromatic k-clique). Show that the former is at least one larger.

Remark This is rather an *existence proof*. Beginning of the "probabilistic method".

Number of RED/BLUE-coloring of $E(K_n)$: $2^{\binom{n}{2}}$.

Number of those containing monochromatic k-clique: For a fixed k-element subset $K \subseteq V(K_n)$, the number of RED/BLUE-coloring of $E(K_n)$ which color K fully RED or or fully BLUE: $2 \cdot 2^{\binom{n}{2}-\binom{k}{2}}$.

$\Longrightarrow$ Number of "bad colorings": $\leq \binom{n}{k} \cdot 2^{\binom{n}{2}-\binom{k}{2}+1}$.

If this is strictly less than $2^{\binom{n}{2}}$ there must be a coloring with no monochromatic k-clique.
That is the case if

$$\binom{n}{k} < 2^{\binom{k}{2}-1}.$$

Certainly true for

$$n \leq \sqrt{2}^k.$$

□

Another **$1000 dollar question:** Prove that

$$\lim_{k\to\infty} \sqrt[k]{R(k)}$$

exists.

Generalization of Ramsey's Theorem ...

... for more colors, hypergraphs.

A pair $(X, \mathcal{F})$, where $\mathcal{F} \subseteq 2^X$ is called a hypergraph.

A hypergraph $(X, \mathcal{F})$ is called t-uniform if all of its members have size t, that is, $\mathcal{F} \subseteq \binom{X}{t}$.

Graph: 2-uniform hypergraph.

Notation: $K_n^{(t)}$ is the complete t-uniform hypergraph on n vertices. $V(K_n^{(t)}) = [n]$, $E(K_n^{(t)}) = \binom{[n]}{t}$

Let $R^{(t)}(k_1, \ldots, k_r)$ be the smallest integer n such that every r-coloring of $\binom{[n]}{t}$ contains a monochromatic copy of $K_{k_i}^{(t)}$ for some i, $1 \leq i \leq r$ (i.e., a set $X \subseteq [n]$ of vertices, $|X| = k_i$, such that every member of $\binom{X}{t}$ has color i).

Ramsey's Theorem For every $r, t \geq 1$, and $k_1, \ldots, k_r \geq 1$, we have $R^{(t)}(k_1, \ldots, k_r) < \infty$

Proof of the General Ramsey Theorem

Fix $r \geq 1$.

Prove by induction on t.
$R^{(1)}(k_1, \ldots k_r) = k_1 + \cdots + k_r - r + 1$.
Assume $t > 1$.

Prove by induction on $k_1 + \ldots + k_r$.
$R^{(t)}(k_1, \ldots, k_{i-1}, 1, k_{i+1}, \ldots, k_r) = 1$.
Assume $\min\{k_1, \ldots, k_r\} > 1$.

Let $n = 1 + R^{(t-1)}(K_1, \ldots, K_r)$,
where $K_i = R^{(t)}(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$
Let $c : E(K_n^{(t)}) \to [r]$ be an arbitrary r-coloring.

Fix an arbitrary vertex, say $n \in [n]$.
Create r-coloring c' of $\binom{[n-1]}{t-1}$: $c'(S) = c(S \cup \{n\})$

Then there is a color i and subset $X \subseteq [n-1]$, $|X| = K_i$, such that $i = c'(S) = c(S \cup \{n\})$ for all $S \in \binom{X}{t-1}$.

Since $K_i = R^{(t)}(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$ there are two cases:

Case 1. For color i, we have a subset $Y \subseteq X$, $|Y| = k_i - 1$, such that all t-tuples are of color i. Since $Y \subseteq X$, all t-tuples of $Y \cup \{n\}$ are of color i.

Case 2. There exist color $j \neq i$ and a subset $Y \subseteq X$, $|Y| = k_j$ such that all t-tuples of Y have color j. $\square$

Application to combinatorial geometry

"**Happy Ending Problem**" (Eszter Klein, 1935) Let $M(k)$ be the smallest such integer M that from any set of M poins on the plane in general position (with no three on a line), there exist k that form a convex k-gon.
Is $M(k) < \infty$?

$M(2) = 2, M(3) = 3, M(4) = 5, M(5) = 9$

Is $M(k) = 2^{k-2} + 1$? We don't know...

Theorem (Erdős-Szekeres, 1935) $M(k) \leq R(k, 5; 4)$ for every k.

Proof. Let P be a set of $n = R(k, 5; 4)$ points in the plane. Define a two-coloring c of the 4-tuples of P:

$$c(\{a, b, c, d\}) = \begin{cases} \text{RED} & a, b, c, d \text{ form a convex 4-gon} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

Claim 1 Among five points in general position there is always four that are in convex position.

Claim 2 If any four points of a k-element point set is in convex position, then the k points are in convex position.

According to Claim 1, there is no BLUE $K_5^{(4)}$ in c, so by the definition of n there is a RED $K_k^{(4)}$. These k points form a convex k-gon by Claim 2. $\square$