Van der Waerden’s Theorem

An $r$-coloring of a set $S$ is a function $c : S \to [r]$. A set $X \subseteq S$ is called monochromatic if $c$ is constant on $X$.

Let $IN$ be two-colored. Is there a monochromatic 3-AP?

Roth’s Theorem says: YES, in the larger of the two color classes.

A weaker statement, not specifying in which color the 3-AP occurs:

**Proposition** In every two-coloring of $\left[2 \cdot (5 \cdot (2^5 + 1))\right]$ there is a monochromactic 3-AP.
What if we want a longer arithmetic progression? Can we color the integers with two colors such that there is no monochromatic 4-AP? Szemerédi’s Theorem says NO.

How far must we color the integers to find an AP of length 4? Or $k$?

In order to prove something about this, we introduce more colors.

$W(r, k)$ is the smallest integer $w$ such that any $r$-coloring of $[w]$ contains a monochromatic $k$-AP.

**Theorem** (van der Waerden, 1927) For every $k, r \geq 1$, $W(r, k) < \infty$.

**Remark** Consequence of Szemerédi’s Theorem.
Proof of Van der Waerden’s Theorem

Induction on $k$, the following statement:
“For all $r \geq 1$, $W(r, k) < \infty$”

$W(r, 1) = 1$
$W(r, 2) = r + 1$
$W(r, 3) =$?

Suppose $W(r, k) < \infty$ for every $r \geq 1$.
Let us find an upper bound on $W(r, k + 1)$ in terms of these numbers.

$W(1, k + 1) = k + 1$

$W(2, k + 1) \leq 2 \cdot (2W(2, k)) \cdot W(2^{2W(2,k)}, k)$

$W(3, k + 1) \leq 2 \cdot 2 \cdot 2W(3, k) \cdot W(3^{2W(3,k)}, k)$
$\cdot W(3^{2 \cdot (2W(3,k))} \cdot W(3^{2W(3,k)}, k), k)$
For general $r$, define the (kind of fast growing) function $L_r : \mathbb{IN} \rightarrow \mathbb{IN}$,

$$L_r(x) = xW(r^x, k).$$

Then

$$W(r, k + 1) \leq 2L_r(\ldots 2L_r(2L_r(2L_r(1))))^{r\text{-times}}.$$ 

We prove by induction on $i$, that no matter how the first $x_i = 2L_r(\ldots 2L_r(2L_r(2L_r(1))))^{i\text{-times}}$ integers are colored with $r$ colors, there exists $i$ monochromatic $k$-APs $a^{(j)}, a^{(j)} + d_j, \ldots, a^{(j)} + (k - 1)d_j, 1 \leq j \leq i$, each in different colors, such that $a^{(j)} + kd_j$ is the very same integer $a$ for each $j, 1 \leq j \leq i$.

Divide $[L_r(x_i)]$ into blocks of $x_i$ integers. There are $r^{x_i}$ ways to $r$-color a block. By the definition of $W(r^{x_i}, k)$, there is a $k$-AP of blocks with the same coloring pattern.

Let $c_j$ be the color of the monochromatic $k$-AP $a^{(j)}, a^{(j)} + d_j, \ldots, a^{(j)} + (k - 1)d_j$, for $1 \leq j \leq i$. 
Case 1. If the color of \( a = a^{(j)} + kd_j \) is one of these colors then there is a \((k + 1)\)-AP in this color and we are done.

Case 2. Otherwise the copies of \( a \) in the \( k \) blocks forms a monochromatic \( k \)-AP of color \( c_{i+1} \neq c_j \), \( 1 \leq j \leq i \). We can form monochromatic \( k \)-APs in the other colors \( c_j \): Take the copy of \( a^{(j)} + (l - 1)d_j \) from the \( l^{th} \) block.

These \( i + 1 \) \( k \)-APs are monochromatic of \( i + 1 \) distinct colors and would be continued in the same \((k + 1)^{st}\) element. This element is certainly less than \( 2L_r(x_i) \).

After the \( r^{th} \) iteration the colors run out, Case 2 cannot occur, and we have a monochromatic \((k + 1)\)-AP. \( \Box \)
Turán-type questions

We are looking for a substructure of a given size.

Turán-type problems: How large fraction of the structure will surely contain a given substructure?

Most natural special case: we are looking for a smaller "copy" of the structure itself.

Turán’s Theorem
Structure: $E(K_n)$
Substructure: $E(K_k)$
Statement:
$F \subseteq E(K_n), \ |F| \geq \left(1 - \frac{1}{k-1}\right) \binom{n}{2} \Rightarrow F \supseteq E(K_k)$

Szemerédi’s Theorem
Structure: $[n]$
Substructure: $k$-AP
Statement: $S \subseteq [n], \ |S| \geq \frac{n}{f(n)} \Rightarrow S$ contains a $k$-AP (for some function $f : \mathbb{N} \rightarrow \mathbb{N}, \ f(n) \rightarrow \infty.$)
Ramsey-type questions

Ramsey-type problems: How large should the structure be such that in any given \( r \)-coloring there is a given substructure that is monochromatic?

**Van der Waerden’s Theorem** (Counterpart of Szemerédi’s Theorem)
Structure: \([n]\)
Substructure: \(k\)-AP
Statement: If \( n \) is large enough, then there is a monochromatic \( k \)-AP in any \( r \)-coloring of \([n]\)

**Ramsey’s Theorem** (Counterpart of Turán’s Theorem)
Structure: \(E(K_n)\)
Substructure: \(E(K_k)\)
Statement: If \( n \) is large enough, then there is a monochromatic \( E(K_k) \) in any \( r \)-coloring of \( E(K_n) \)
Ramsey’s Theorem

**Proposition** In any RED/BLUE-coloring of $E(K_6)$ there is either a RED $K_3$ or a BLUE $K_3$

Let $R(k)$ be the smallest integer $R$ such that any two-coloring of $E(K_R)$ contains a monochromatic copy of $K_k$.

**Proposition** $R(3) = 6$

To prove the existence of $R(k)$ in general we need a bit more general notion.

Let $R(k, l)$ be the smallest integer $R$ such that any RED/BLUE-coloring of $E(K_R)$ contains a RED copy of $K_k$ or a BLUE copy of $K_l$.

**Ramsey’s Theorem** For any $k, l \geq 1$, we have

$$R(k, l) < \infty.$$
Proof of Ramsey’s Theorem

Induction on $k + l$.
$R(k, 1) = R(1, l) = 1$
$R(k, 2) = k, R(2, l) = l$

Claim $R(k, l) \leq R(k - 1, l) + R(k, l - 1)$

Proof. Let $R = R(k - 1, l) + R(k, l - 1)$ and let $c$ be an arbitrary RED/BLUE-coloring of $E(K_R)$.

Let $x \in V(K_R)$ arbitrary.
Define $N_{blue} = \{v \in V(K_R): xv \text{ is BLUE}\}$ and $N_{red} = \{v \in V(K_R): xv \text{ is RED}\}$

Since $|N_{blue}| + |N_{red}| + 1 = R$ we have either $|N_{blue}| \geq R(k, l - 1)$ or $|N_{red}| \geq R(k - 1, l)$.

If $|N_{blue}| \geq R(k, l - 1)$ then there is a RED $K_k$ or a BLUE $K_{l-1}$ which together with $x$ forms a BLUE $K_l$.

If $|N_{red}| \geq R(k - 1, l)$ then there is a BLUE $K_l$ or a RED $K_{k-1}$ which together with $x$ forms a RED $K_k$. □
Corollaries

**Corollary** \( R(k, l) \leq \binom{k+l-2}{k-1} \)

*Proof.* By induction on \( k + l \).

\[
R(k, l) \leq R(k - 1, l) + R(k, l - 1) \\
\leq \binom{k + l - 3}{k - 2} + \binom{k + l - 3}{k - 1} = \binom{k + l - 2}{k - 1}
\]

**Corollary** \( R(k) < 4^k \)

*$1000 dollar question:* How large is \( R(k) \)?
Lower bounds

**Lower bound** The “Turán-graph coloring” gives 
\[(k - 1)^2 < R(k).\]

“Construction” (Erdős, 1951) \[\sqrt{2^k} < R(k)\]

**Strategy:** Count all two-colorings of \(E(K_n)\) as well as the “bad” ones (the ones that contain a monochromatic \(k\)-clique). Show that the former is at least one larger.

**Remark** This is rather an existence proof. Beginning of the “probabilistic method”.

Number of RED/BLUE-coloring of \(E(K_n)\): \[2^{\binom{n}{2}}.\]

Number of those containing monochromatic \(k\)-clique:
For a fixed \(k\)-element subset \(K \subseteq V(K_n)\), the number of RED/BLUE-coloring of \(E(K_n)\) which color \(K\) fully RED or or fully BLUE: \[2 \cdot 2^{\binom{n}{2}} - \binom{k}{2}.\]
Number of “bad colorings”: \[ \leq \binom{n}{k} \cdot 2^{\binom{n}{2} - \binom{k}{2} + 1}. \]

If this is strictly less than \(2^{\binom{n}{2}}\) there must be a coloring with no monochromatic \(k\)-clique.
That is the case if
\[ \binom{n}{k} < 2^{\binom{k}{2} - 1}. \]
Certainly true for
\[ n \leq \sqrt{2^k}. \]

Another \$1000\ dollar question: Prove that
\[ \lim_{k \to \infty} k^{\frac{1}{k}} R(k) \]
exists.
Generalization of Ramsey’s Theorem ... ____

... for more colors, hypergraphs.

A pair \((X, \mathcal{F})\), where \(\mathcal{F} \subseteq 2^X\) is called a hypergraph.

A hypergraph \((X, \mathcal{F})\) is called \(t\)-uniform if all of its members have size \(t\), that is, \(\mathcal{F} \subseteq \binom{X}{t}\).

Graph: 2-uniform hypergraph.

Notation: \(K_n^{(t)}\) is the complete \(t\)-uniform hypergraph on \(n\) vertices. \(V(K_n^{(t)}) = [n], \ E(K_n^{(t)}) = \binom{[n]}{t}\)

Let \(R^{(t)}(k_1, \ldots, k_r)\) be the smallest integer \(n\) such that every \(r\)-coloring of \(\binom{[n]}{t}\) contains a monochromatic copy of \(K_{k_i}^{(t)}\) for some \(i, 1 \leq i \leq r\) (i.e., a set \(X \subseteq [n]\) of vertices, \(|X| = k_i\), such that every member of \(\binom{X}{t}\) has color \(i\)).

**Ramsey’s Theorem** For every \(r, t \geq 1\), and \(k_1, \ldots, k_r \geq 1\), we have \(R^{(t)}(k_1, \ldots, k_r) < \infty\)
Proof of the General Ramsey Theorem

Fix $r \geq 1$.

Prove by induction on $t$.

$R^{(1)}(k_1, \ldots, k_r) = k_1 + \cdots + k_r - r + 1$.

Assume $t > 1$.

Prove by induction on $k_1 + \ldots + k_r$.

$R^{(t)}(k_1, \ldots, k_{i-1}, 1, k_{i+1}, \ldots, k_r) = 1$.

Assume $\min\{k_1, \ldots, k_r\} > 1$. 
Let \( n = 1 + R^{(t-1)}(K_1, \ldots, K_r) \),
where \( K_i = R^{(t)}(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r) \)
Let \( c : E(K_n^{(t)}) \to [r] \) be an arbitrary \( r \)-coloring.

Fix an arbitrary vertex, say \( n \in [n] \).
Create \( r \)-coloring \( c' \) of \( ([n-1]_t) : c'(S) = c(S \cup \{n\}) \)

Then there is a color \( i \) and subset \( X \subseteq [n-1] \), \( |X| = K_i \), such that \( i = c'(S) = c(S \cup \{n\}) \) for all \( S \in X_{t-1} \).

Since \( K_i = R^{(t)}(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r) \)
there are two cases:

**Case 1.** For color \( i \), we have a subset \( Y \subseteq X \), \( |Y| = k_i - 1 \), such that all \( t \)-tuples are of color \( i \). Since \( Y \subseteq X \), all \( t \)-tuples of \( Y \cup \{n\} \) are of color \( i \).

**Case 2.** There exist color \( j \neq i \) and a subset \( Y \subseteq X \), \( |Y| = k_j \) such that all \( t \)-tuples of \( Y \) have color \( j \). \( \square \)
Application to combinatorial geometry

“Happy Ending Problem” (Eszter Klein, 1935) Let $M(k)$ be the smallest such integer $M$ that from any set of $M$ points on the plane in general position (with no three on a line), there exist $k$ that form a convex $k$-gon.

Is $M(k) < \infty$?

$M(2) = 2, M(3) = 3, M(4) = 5, M(5) = 9$

Is $M(k) = 2^{k-2} + 1$? We don’t know...

**Theorem** (Erdős-Szekeres, 1935) $M(k) \leq R(k, 5; 4)$ for every $k$.

**Proof.** Let $P$ be a set of $n = R(k, 5; 4)$ points in the plane. Define a two-coloring $c$ of the 4-tuples of $P$:

$$c(\{a, b, c, d\}) = \begin{cases} 
\text{RED} & \text{a, b, c, d form a convex 4-gon} \\
\text{BLUE} & \text{otherwise}
\end{cases}$$
Claim 1 Among five points in general position there is always four that are in convex position.

Claim 2 If any four points of a $k$-element point set is in convex position, then the $k$ points are in convex position.

According to Claim 1, there is no BLUE $K_5^{(4)}$ in $c$, so by the definition of $n$ there is a RED $K_k^{(4)}$. These $k$ points form a convex $k$-gon by Claim 2.  

\qed