Line graphs and edge coloring

A k-edge-coloring of a multigraph G is a labeling $f : E(G) \to S$, where $|S| = k$. The labels are called colors; the edges of one color form a color class. A k-edge-coloring is proper if incident edges have different labels. A multigraph is k-edge-colorable if it has a proper k-edge-coloring.

The edge-chromatic number (or chromatic index) of a loopless multigraph G is

$$\chi'(G) := \min\{k : G \text{ is } k\text{-edge-colorable}\}.$$

A multigraph G is k-edge-chromatic if $\chi'(G) = k$.

Remarks. $\chi'(G) = \chi(L(G))$, so

$$\begin{aligned} \Delta(G) &\leq \omega(L(G)) \\ &\leq \chi'(G) \leq \Delta(L(G)) + 1 \\ &\qquad\qquad \leq 2\Delta(G) - 1 \end{aligned}$$

Vizing's Theorem

Example. K_{2n}

Theorem. (König, 1916)
For a bipartite multigraph G, $\chi'(G) = \Delta(G)$

Proposition. $\chi'(Petersen) = 4$.

Theorem. (Vizing, 1964) For a simple graph G,

$$\chi'(G) \leq \Delta(G) + 1.$$

Generalization. If the maximum edge-multiplicity in a multigraph G is $\mu(G)$, then $\chi'(G) \leq \Delta(G) + \mu(G)$
Example. Fat triangle; $\chi'(G) = \Delta(G) + \mu(G)$.

Proof of Vizing's Theorem (A. Schrijver)

Induction on $n(G)$.

If $n(G) = 1$, then $G = K_1$; the theorem is OK.

Assume $n(G) > 1$. Delete a vertex $v \in V(G)$. By induction $G - v$ is $(\Delta(G) + 1)$-edge-colorable.

Why is G also $(\Delta(G) + 1)$-edge-colorable?

We prove the following

Stronger Statement. Let $k \geq 1$ be an integer. Let $v \in V(G)$, such that

- $d(v) \leq k$,
- $d(u) \leq k$ for every $u \in N(v)$, and
- $d(u) = k$ for at most one $u \in N(v)$.

Then
$G - v$ is k-edge-colorable $\Rightarrow$ G is k-edge-colorable.

Proof of the Stronger Statement I

Induction on k (!!!)

For $k = 1$ it is OK.

W.l.o.g. $d(u) = k - 1$ for every $u \in N(v)$, except for *exactly one* $w \in N(v)$ for which $d(w) = k$.

Let $f : E(G - v) \to \{1, \ldots, k\}$ be a proper k-edge-coloring of $G - v$, which minimizes*

$$\sum_{i=1}^{k} |X_i|^2.$$

Here $X_i := \{u \in N(v) : \ u \text{ is missing color } i\}$.

*I.e., we choose the coloring so the $|X_i|$s "as equal as possible".

Proof of the Stronger Statement II

Case 1. There is an i, with $|X_i| = 1$. Say $X_k = \{u\}$.

Let $G' = G - uv - \{xy : f(xy) = k\}$.
Apply the induction hypothesis for G' and $k - 1$.

Case 2. $|X_i| \neq 1$ for every $i = 1, \ldots, k$.

Then

$$\sum_{l=1}^{k} |X_l| = 2d(v) - 1 < 2k.$$

So there are colors i with $|X_i| = 0$ and
j with $|X_j| \geq 3$.

Let $H \subseteq G$ be subgraph spanned by the edges of color i and j.
Switch colors i and j in a component C of H, where $C \cap X_j \neq \emptyset$.
This reduces $\sum_{l=1}^{k} |X_l|^2$, a contradiction. □