Hamiltonian cycles

A spanning cycle is called a Hamiltonian cycle. A graph is called Hamiltonian if it contains a Hamiltonian cycle.

Example $K_{m,n}$

A spanning path is called a Hamiltonian path.

Example. Petersen graph.is not Hamiltonian

Arthur and Merlin – a touch of complexity

A: Show me a pairing, so my 150 knights can marry these 150 ladies!
M: Not possible!
A: Why?
M: Here are these 93 ladies and 58 knights, none of them are willing to marry each other.
A: Alright, alright ...

A: Seat my 150 knights around the round table, so that neighbors don't fight!
M: Not possible!
A: Why?
M: It will take me forever to explain you.
A: I don't believe you! Into the dungeon!

A YES/NO-problem problem is in the class *NP:* The answer **YES** can be checked “efficiently”
”efficiently”: within a time, which is polynomial in the size of the input

In other words:

- there is a ”certificate”, which a computer (i.e., Arthur, i.e., a polynomial time algorithm) can verify within a reasonable time

Note: the certificate can be provided by an all-powerful supercomputer (i.e., Merlin)

Examples:

“Does this bipartite graph have a perfect matching?” (provide perfect matching)

“Does this bipartite graph have **no** perfect matching?” (provide vertex cover of size **less** than $n/2$; certificate exists because of König’s Theorem)

“Does this graph have a Hamilton cycle?” (provide Hamilton cycle)

Merlin’s Pech: “Does this graph have **no** Hamilton cycle?” is not (known to be) in NP

A YES/NO-problem is in the class *co-NP:* The answer **NO** can be checked efficiently

Properties having a ”good” characterization or a min/max theorem are both in NP and co-NP

Examples:

- ”Is this graph 2-colorable?” (NP: provide a 2-coloring; co-NP: provide an odd cycle)
- ”Is this graph Eulerian?” (NP: provide an ordered list of the edges for an Eulerian circuit; co-NP: provide a vertex with an odd degree; co-NP certificate **exists** because of Euler’s Theorem)
- ”Does this graph have a perfect matching?” (NP: provide a perfect matching; co-NP: provide a subset S whose deletion creates more than $|S|$ odd components; co-NP certificate **exists** because of Tutte’s Theorem)
- ”Is this graph k-connected?” (NP: for each two vertices $x, y \in V(G)$ provide a list of k internally disjoint x, y-path; co-NP: provide a cut-set of size less than k; NP-certificate **exists** because of Menger’s Theorem)

A YES/NO-problem is in the class *P*: The answer can be **found** efficiently (i.e., there is a polynomial time algorithm to actually obtain the certificate (i.e., no need for Merlin))

Of course: $P \subseteq NP \cap co\text{-}NP$

Often: Problems in $NP \cap co\text{-}NP$ are also in P

However: People think $P \neq NP \cap co\text{-}NP$

We don't know: problem of "Is there a factor less than k?"

People also think: $P \neq NP$ (1,000,000 US dollars)

We don't know: Hamiltonicity, 3-colorability, $\Delta(G)$-edge-colorability, k-independence set,

A necessary condition

Proposition. If G is Hamiltonian, then for every $S \subseteq V$, $c(G - S) \leq |S|$.

$c(H)$ is the number of components of graph H.

Remark. A graph G is t-tough if $|S| \geq tc(G - S)$ for every cut-set $S \subseteq V(G)$. The toughness of G is the maximum t such that G is t-tough.

The toughness of the Petersen graph is $4/3$.

Toughness Conjecture (Chvátal, 1973) There is a value t such that every graph of toughness at least t is Hamiltonian.

A stronger version of this conjecture stated that every 2-tough graph is Hamiltonian. This turned out to be false. Bauer-Broersma-Veldman (2000) constructed a family of non-Hamiltonian graphs with toughness approaching $9/4$.

No toughness larger than 1 is necessary, since C_n is Hamiltonian.

A sufficient degree condition

Theorem. (Dirac,1952) If $G = (V, E)$ is a simple graph with $n(G) \geq 3$ and $\delta(G) \geq \lceil n(G)/2 \rceil$, then G is Hamiltonian.

Proof. Take a counterexample G with the *most number of edges*. So addition of any edge to G creates a Hamilton cycle.

Take a Hamilton path $v_1, \ldots, v_n$, where $v_1 v_n \notin E$.

Let $N^-(v_1) = \{v_{j-1} : v_j v_1 \in E\}$.

Since $|N(v_n)|, |N^-(v_1)| \geq \frac{n}{2}$
and $N(v_n), N^-(v_1) \subseteq V \setminus \{v_n\}$,
there is a vertex $v_i \in N(v_n) \cap N^-(v_1)$

Hamilton cycle in G: $v_n v_i v_{i-1} \ldots v_1 v_{i+1} \ldots v_n$. □

Remark. Dirac's Theorem is best possible. Non-Hamiltonian graphs with minimum degree $\lceil \frac{n}{2} \rceil - 1$:
- $K_{\lceil (n+1)/2 \rceil}$ and $K_{\lfloor (n+1)/2 \rfloor}$ sharing a vertex.
- For odd n an alternativet example is $K_{\frac{n-1}{2}, \frac{n+1}{2}}$

Chvátal's degree condition

Theorem. (Bondy-Chvátal, 1976) G is Hamiltonian iff its Hamiltonian closure $C(G)$ is Hamiltonian.

The Hamiltonian closure $C(G)$ of a graph G is the graph with vertex set $V(G)$ obtained from G by iteratively adding edges joining pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains.

Lemma. The closure of G is well-defined.

Lemma. (Ore, 1960) Let $u, v \in V(G)$, such that $d(u) + d(v) \geq n(G)$. Then G is Hamiltonian iff $G + uv$ is Hamiltonian.

A sufficient condition through connectivity

Theorem. (Erdős-Chvátal, 1972) If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian. (Unless $G = K_2$)

Proof. Let $k = \kappa(G) > 1$. Let $C = (v_1, \ldots v_\ell)$ be the longest cycle.

$\delta(G) \geq k \Rightarrow length(C) \geq k + 1$

Let H be a component of $G - C$.

Let $v_{i_1}, \ldots v_{i_k} \in V(C)$ be vertices with an edge to $V(H)$. Then:

- $U = \{v_{i_1+1}, \ldots, v_{i_k+1}\}$ is independent
- No edge between U and $V(H)$.

$\Rightarrow \alpha(G) \geq k + 1$. □