Exercise Sheet 11

Tibor Szabó
Discrete Mathematics II, Winter 2011/12
Due date: January 24th (Tuesday) by 12:30, at the beginning of the exercise session.

Problem 1. Show that for any integer $k \geq 1$ there exists an integer $N = N(k)$, such that in any $N \times N$ 0/1-matrix there exists a principal $k \times k$ submatrix in which all elements above the diagonal are the same and all elements below the diagonal are the same.

Problem 2. Let G be a graph define as follows:

$$\begin{aligned} V(G) &= \binom{[k]}{3} \\ E(G) &= \{AB : |A \cap B| = 1\} \end{aligned}$$

Prove that both $\alpha(G)$ and $\omega(G)$ is at most k.

(Give an independent *combinatorial* proof (don't apply any general theorems you will learn the coming week.))

Remark. This construction of Zs. Nagy (1973) gives an explicit constructive lower bound of cubic order for the symmetric Ramsey number $R(k+1, k+1)$. This was the first successful attempt to exceed the quadratic lower bound (of k^2) provided by the Turán graph. We will discuss much more of explicit combinatorial constructions next semester in the Discrete Mathematics III course. Recall that the exponential lower bound of Erdős which was mentioned at the lecture is not constructive. We will prove it using the probabilistic method.

Problem 3. Prove Dilworth's Theorem using König's min-max Theorem about the maximum size matching in bipartite graph.

Problem 4. Let $SF(\ell, k)$ be the smallest integer N, such that every family of N sets of size ℓ contains a sunflower with k petals.

(a) Show that $SF(2,3) = 7$.

(b) Show that for every even ℓ, we have $SF(\ell, 3) > \sqrt{6}^{\ell}$

Problem 5.

(a) Show that any two-coloring of the integers contains a monochromatic solution of the equation $x + 2y = z$. (*Hint:* Try to make use of van der Waerden's Theorem in your proof)

(b) Construct a coloring of the positive integers so there is no monochromatic solution to the equation $x + y = 3z$. (*Hint:* 4-color according to the mod 5 remainder after the largest power of 5 is factored out.)

(c) Let $a_1, \dots, a_n \in \mathbb{Z}$ be constants. Show that there is a coloring of the positive integers with finitely many colors such a way that $a_1x_1 + \cdots + a_nx_n = 0$ does **not** have monochromatic solution $(x_1, \dots, x_n)$ if and only if there is a subset of the coefficients which sum up to 0. (That is, there exists $I \subseteq [n]$, such that $\sum_{i \in I} a_i = 0$.)

Remark Van der Waerden's Theorem and Problem 4 on the previous sheet are both special cases of (c); though the VdW Thm is used in the proof above, while Problem 4 on Sheet 10 could be proved in a much simpler way.
Parts (a) and (b) are instructive special cases which should lead you to the general solution.)

Problem 6. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of subsets of $[n]$, such that for every $i \neq j$, $F_i \not\subseteq F_j$, $F_i \cap F_j \neq \emptyset$, $F_i \cup F_j \neq [n]$. Prove that

$$m \leq \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

(*Hint:* You might want to use Hall's Theorem to reduce to the case when $\mathcal{F}$ is uniform.)