Exercise Sheet 12

Tibor Szabó
Discrete Mathematics II, Winter 2011/12
Due date: January 31st (Tuesday) by 12:30, at the beginning of the exercise session.

Definition A family $A_1, \ldots, A_m \subseteq [n]$ is said to satisfy the Mod$k$-town rules

(1) $|A_i| \not\equiv 0 \pmod{k}$ for every i

(2) $|A_i \cap A_j| \equiv 0 \pmod{k}$ for every $i \neq j$

Problem 1. Prove that if a family $\mathcal{F} \subseteq 2^{[n]}$ satisfies the Mod6-town rules, then $|\mathcal{F}| \leq 2n$.

Problem 2. Prove that if a family $\mathcal{F} \subseteq 2^{[n]}$ satisfies the Modp^k-town rules for some prime p, then $|\mathcal{F}| \leq n$. (*Hint:* Show linear independence of the characteristic vectors over $\mathbb{Q}$.)

Problem 3. In the Reverse Oddtown rules the words "odd" and "even" of the Oddtown rules are switched.

(a) Prove that no more than n clubs can be formed under the Reverse Oddtown rules in a town of n citizens.

(b) Determine the maximum for each n

Hint: The answer depends on parity.

Problem 4. (a) Prove that $\chi(G_2) \geq 4$ (*Hint:* Construct a graph on 7 vertices that is a subgraph of G_2 and is not three-colorable.)

(b) Prove that $\chi(G_2) \leq 7$

(c) Prove that $\chi(G_n) \leq n^{n/2} \cdot 2^n$. (*Hint*: Divide the space into cubes of diameter 3/4).