Exercise Sheet 14

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Discrete Mathematics II, Winter 2011/12
Due date: February 15th (Wednesday) by 8:30, at the beginning of the exercise session.

Problem 1. Let G be a plane graph where every face is a triangle. Prove that for any 3-coloring of the vertices of G there are an even number of faces whose vertices are colored with three different colors.

Problem 2. For some positive integer k, a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is called a *k-CNF formula* if f is the *AND* of an arbitrary (finite) number of clauses, each *clause* is the *OR* of exactly k literals of k distinct variables (where each *literal* is either a variable or its negation). We say that the formula is *satisfiable* if f is not constant 0.
(For example, $(x_1 \vee x_3 \vee \bar{x}_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_5) \wedge (\bar{x}_1 \vee x_4 \vee x_5)$ is a 3-CNF formula of the variables x_1, x_2, x_3, x_4, and x_5. For example f evaluates to 0 (that is, "`false`") with the substitution $(0,0,0,1,1)$ and to 1 (that is, "`true`") with the substituition $(1,0,1,1,0)$. Hence f is satisfiable, because it has at least one substituition which evaluates to 1.)

A k-CNF formula in which each variable appears in at most s clauses is called a (k,s)-CNF formula. (The example above is a $(3,2)$-CNF formula because no variable appears in three clauses. Let $f(k)$ be the largest integer s such that every (k,s)-CNF formula is satisfiable.

(i) Show that $2^k > f(k)$ for every $k \geq 1$.
(ii) Show that $f(k) \geq k$ for every $k \geq 1$.
(iii) Show that $f(k) \geq \left\lfloor \frac{2^k}{ke} \right\rfloor - 1$ for all $k \geq 1$.