Exercise Sheet 6

Tibor Szabó

Discrete Mathematics II, Winter 2011/12

Due date: November 29th (Tuesday) by 12:30, at the beginning of the exercise session.

Problem 1. Formulate and prove a mathematically precise statement expressing the following: The stable matching produced by the Proposal Algorithm with the men doing the proposing is the best possible stable matching for the men.

Problem 2. A total coloring of G assigns a color to each vertex and to each edge such that the colored objects have different colors when they are adjacent or incident. Prove that G has a total coloring with at most $\chi'_l(G) + 2$ colors.
Remark. The Total Coloring Conjecture states that every simple graph G has a total coloring with at most $\Delta(G) + 2$ colors. By the above exercise, the List Coloring Conjecture would imply a total coloring with $\Delta(G) + 3$ colors.)

Problem 3. Prove that every connected graph with an even number of edges can be decomposed into P_3s. (*Hint:* One option is to use Tutte's Theorem.)

Problem 4. Let G be a graph and $M = \{v \in V(G) : d(v) = \Delta(G)\}$ be the set of vertices with maximum degree. Show that if the induced subgraph $G[M]$ is a forest, then $\chi'(G) = \Delta(G)$.

Problem 5. The Cartesian product $G \square H$ of two graphs G and H can informally be thought of as taking $n(H)$ vertex-disjoint copies of the graph G and introducing a perfect matching between each two copies of G which correspond to two adjacent vertices of H (and these perfect matchings connect the identical vertices in the two copies of G). (For example the d-dimensional hypercube graph Q_d is the Cartesian product of two smaller cubes whose dimension sums up to d: $Q_d \simeq Q_k \square Q_{d-k}$ for any $1 \le k \le d-1$.)
Formally, let $V(G \square H) = V(G) \times V(H)$ and $E(G \square H) = \{\{(g_1, h_1), (g_2, h_2)\} : (\{g_1, g_2\} \in E(G) \text{ and } h_1 = h_2) \text{ or } (g_1 = g_2 \text{ and } \{h_1, h_2\} \in E(H))\}$

Prove that if G and H are simple graphs with at least one edge, then $\chi'(H) = \Delta(H)$ implies $\chi'(G \square H) = \Delta(G \square H)$.