Exercise Sheet 8

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Discrete Mathematics II, Winter 2011/12
Due date: January 3rd (Tuesday) by 12:30, at the beginning of the exercise session.

Problem 1. Prove that a k-chromatic triangle-free graph must have at least $\binom{k+1}{2} - 1$ vertices.
Remark: Mycielski's construction gives an example with exponentially many ($\sim 2^k$) vertices.

Problem 2. Let G be the graph on vertex set $[4]$ with edges $12, 13, 14, 23$. Determine $ex(n, G)$ for every n.

Problem 3.

(a) Let $G = (V, E)$ be a graph with average vertex degree $\bar{d}$. Show that there exists a subgraph H in G with $\delta(H) \geq \bar{d}/2$.

(b) Let T be a tree on k vertices. Prove that $\frac{n(k-2)}{2} \leq \mathrm{ex}(n, T) < nk$, for any $n \in \mathbb{N}$ divisible by $k - 1$.

Remark: The lower bound is conjectured to be tight.

Problem 4. Consider the finite field $\mathbb{F}_p$ for some prime number p and define a graph $G_p = (V_p, E_p)$ on the vertex set $V_p = \mathbb{F}_p^2 \setminus \{(0,0)\}$. Two points (x, y) and (x', y') from V_p are connected by an edge in G_p if and only if they are distinct and $xx' + yy' \equiv 1 \pmod{p}$.

a) Prove that G_p does not contain $K_{2,2}$ as a subgraph.

b) Show that $|E_p| \geq \frac{1}{2}(p-1)(p^2-1)$.

c) Conclude that $\mathrm{ex}(n, K_{2,2}) = \Omega(n^{3/2})$.

Remark. Together with Erdős' Theorem from the lecture, we have $\mathrm{ex}(n, K_{2,2}) = \Theta(n^{3/2})$.

Problem 5. Prove that for any $s, t \in \mathbb{N}$ there is some constant $c = c(s,t) \in \mathbb{R}$ such that $\mathrm{ex}(n, K_{s,t}) \leq cn^{2-1/s}$, for any $n \in \mathbb{N}$.
Hint: Doublecount the subgraphs $K_{1,s}$.

Problem 6.

(i) Prove that the Turán graph $T_{n,r-1}$ is a unique graph which maximizes the sum of the squared degrees (i.e., $\sum_{v\in V(G)} d(v)^2$) among all K_r-free n-vertex graphs G. (Hint: Mimic the proof of Turán's theorem.)

(ii) Prove that the statement of part (a) is no longer generally true if we consider maximizing $\sum_{v\in V(G)} d(v)^4$ instead.