Exercise Sheet 9

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Discrete Mathematics II, Winter 2011/12
Due date: January 10th (Tuesday) by 12:30, at the beginning of the exercise session.

Problem 1.

(*i*) Show that for any set P of n points in $\mathbb{R}^2$ there are at most $O(n^{3/2})$ pairs of points in $\binom{P}{2}$ that have Euclidean distance exactly one. (*Hint:* Show that an appropriately defined *unit distance graph* does not contain $K_{2,3}$ as a subgraph.)
Remark. It is conjectured that the same is true with $n^{1+o(1)}$ instead of $n^{3/2}$. The current best bound is $n^{4/3}$.

(*ii*) Show that for any set P of n points in $\mathbb{R}^3$ there are at most $O(n^{5/3})$ pairs of points in $\binom{P}{2}$ that have Euclidean distance exactly one.

Problem 2. If A, B is ε-regular with $\mathrm{d}(A, B) = \delta$ and $A' \subset A$, $B' \subset B$ satisfy $|A'| \geq \gamma|A|$ and $|B'| \geq \gamma|B|$ for some $\gamma \geq \varepsilon$, then A', B' is $\varepsilon \max\{2, \frac{1}{\gamma}\}$-regular with density somewhere in $[\delta - \varepsilon, \delta + \varepsilon]$.

Problem 3. (Triangle Removal Lemma) For some $\gamma > 0$, a graph $G = (V, E)$ is said to be γ-*far* from having some property $\mathcal{P}$ if one needs to change (add or remove) more than $\gamma\binom{|V|}{2}$ edges in G to make it satisfy $\mathcal{P}$.

Show that for every $\gamma > 0$ there is a $\delta = \delta(\gamma)$ such that any graph $G = (V, E)$ which is γ-far from being triangle-free contains at least $\delta\binom{|V|}{3}$ triangles.

Problem 4. Show that for any $\eta \in (0, 1)$ and $k \in \mathbb{N}$ there exist $\gamma = \gamma(\eta, k)$ and $\delta = \delta(\eta, k)$ with the following property.
Consider a graph $H = (\{v_1, \ldots, v_k\}, E)$ and let $V_1, \ldots, V_k$ be pairwise disjoint vertex sets of some graph G such that V_i, V_j is γ-regular for any $1 \leq i < j \leq k$. Moreover, suppose $\mathrm{d}(V_i, V_j) \geq \eta$, if $\{v_i, v_j\} \in E$ and $\mathrm{d}(V_i, V_j) \leq 1 - \eta$, otherwise. Then at least $\delta \prod_{i=1}^{k} |V_i|$ tuples from $V_1 \times \ldots \times V_k$ span induced copies of H.