Hypergraph Turán numbers I – 4-clique

What would be the smallest meaningful clique to generalize Turán's Theorem for in k-uniform hypergraphs with $k > 2$? It is $K_4^{(3)}$.

Construction Let $3|n$. Partition $V_0 \cup V_1 \cup V_2 = [n]$ with $|V_0| = |V_1| = |V_2| = \frac{n}{3}$. Let $\mathcal{H}$ be 3-uniform:
$E(\mathcal{H}) = \{T : |T \cap V_i| = 1 \text{ for all } i = 0, 1, 2\} \cup$
$\{T : |T \cap V_i| = 2, |T \cap V_{i+1}| = 1 \text{ for some } i = 0, 1, 2\}$

Proposition $\mathcal{H}$ contains no copy of $K_4^{(3)}$.

For an k-uniform hypergraph $\mathcal{K}$, let $ex(n, \mathcal{K})$ be the largest number m such that there exists a $\mathcal{K}$-free k-uniform hypergraph on n vertices with m edges.

Consequence $ex(n, K_4^{(3)}) \geq \frac{5}{9}\binom{n}{3}$

Turán's Conjecture ($1000 dollar question)

$$ex(n, K_4^{(3)}) = |E(\mathcal{H})|$$

Remark If conjecture is true, then there are exponentially many extremal constrcutions (Kostochka).

Hypergraph Turán numbers II — Fano plane

Let $\mathcal{F}$ be the 3-uniform hypergraph defined on $V(\mathcal{F}) = [7]$ with $E(\mathcal{F}) = \{123, 345, 561, 174, 376, 572, 246\}$.

Remark $\mathcal{F}$ is called the "Fano plane" (It is the projective plane over the field $\mathbb{F}_2$). Its sets have the nice property that any two of them interesct in exactly 1 element.

A coloring of the vertices of a hypergraph $\mathcal{H}$ is proper if no edge is monochromatic.

Proposition $\mathcal{F}$ is not properly 2-colarable.

Construction Let $\mathcal{H}$ be the 2-colorable hypergraph with the most edges: Partition $V_1 \cup V_2 = [n]$ with $|V_1| = \lfloor \frac{n}{2} \rfloor$ and $|V_2| = \lceil \frac{n}{2} \rceil$.
$E(\mathcal{H}) = \{T \in \binom{[n]}{3} : T \cap V_i \neq \emptyset$ for both $i = 1, 2\}$

Claim $\mathcal{H}$ contains no copy of $\mathcal{F}$.

Proof. $\mathcal{F}$ is not 2-colorable. □

Theorem (De Caen-Füredi, Keevash-Sudakov, Füredi-Simonovits, 2006) $ex(n, \mathcal{F}) = |E(\mathcal{H})|$

Extremal set theory — the classics I

A family $\mathcal{F}$ of sets is called k-uniform if every member is a k-elements set.

Family $\mathcal{S}$ is a sunflower (or Δ-system) if $A \cap B = \cap_{F \in \mathcal{S}} F$ for every $A, B \in \mathcal{S}$. The set $\cap_{F \in \mathcal{S}} F$ is called the core of the sunflower and $F \setminus \cap_{F \in \mathcal{S}} F$ are its petals.

Theorem (Erdős-Rado) $\mathcal{F}$ is an ℓ-uniform family and $|\mathcal{F}| \geq 2^\ell \ell!$ then $\mathcal{F}$ contains a sunflower with three petals

Construction $X = \{x_1, \ldots, x_\ell, y_1, \ldots, y_\ell\}$
Define $\mathcal{F} = \{F \subseteq X : |F \cap \{x_i, y_i\}| = 1 \text{ for every } i\}$.
$\mathcal{F}$ has no sunflower with three petals and $|\mathcal{F}| = 2^l$.

There are better constructions with C^ℓ members where C is some constant > 2 (HW). But no superexponential construction is known.

The best known upper bound (Kostochka) is slightly smaller than $\ell!$.

\$1000 dollar question: Is there an ℓ-uniform family containing no sunflower with three petals, which has superexponential size (in ℓ)?

Proof. Induction on ℓ. For $\ell = 1$ we can have at most two one-element subsets.

Let $\ell > 1$.
There exist a set X of at most 2ℓ elements that every $F \in \mathcal{F}$ intersect X (Take two disjoint members of $\mathcal{F}$ if they exist, otherwise take any one member of $\mathcal{F}$.)

$\mathcal{F}_x = \{F \setminus \{x\} : F \in \mathcal{F}, x \in F\}$ is an $(\ell-1)$-uniform family containing no sunflower with three petals, for every $x \in X$.

By induction $|\mathcal{F}_x| \leq 2^{\ell-1}(\ell-1)!$ for every $x \in X$.

Then

$$|\mathcal{F}| \leq \sum_{x \in X} |\mathcal{F}_x| \leq |X| \cdot (2^{\ell-1}(\ell-1)!) \leq 2^{\ell}\ell!.$$

Posets

$(P, \leq)$ is a poset if the relation $\leq$ on P is

- reflexive ($a \leq a$ for all $a \in P$)
- antisymmetric ($a \leq b$ and $b \leq a \Rightarrow a = b$)
- transitive ($a \leq b$ and $b \leq c \Rightarrow a \leq c$)

a and b are comparable if $a \leq b$ or $b \leq a$. Otherwise a and b are incomparable.

$C \subseteq P$ is a chain if any two elements are comparable.

$A \subseteq P$ is an antichain if no two elements are comparable.

Extremal set theory — the classics II

The width of a poset is the size of the largest antichain.

$(2^{[n]}, \subseteq)$ is the Boolean poset.

Sperner's Theorem The width of the Boolean poset is $\binom{n}{\lfloor n/2 \rfloor}$.

Reformulation: How many subsets of $[n]$ can be select if it is forbidden to select two sets such that one is subset of the other?

You can select all $\binom{n}{k}$ subsets of a given size k: they certainly satisfy the property.
$k = \left\lfloor \frac{n}{2} \right\rfloor$ maximizes their number.

Sperner's Theorem If $\mathcal{F} \subseteq 2^{[n]}$ is a family of subsets such that for every $A, B \in \mathcal{F}$ we have $A \not\subseteq B$ then

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Permutation method

Proof. Count permutations $\pi \in S_n$ of $[n]$ which have an initial segment from $\mathcal{F}$. Formally, double-count

$$M = |\{(\pi, F) : \pi \in S_n, F \in \mathcal{F}, F = \{\pi(1), \ldots, \pi(|F|)\}\}|$$

For every $F \in \mathcal{F}$ there are $|F|!(n - |F|)!$ permutations $\pi \in S_n$ with $\{\pi(1), \ldots, \pi(|F|)\} = F$. So

$$M = \sum_{F \in \mathcal{F}} |F|!(n - |F|)!.$$

For every $\pi \in S_n$ there is at most one k such that $\{\pi(1), \ldots, \pi(k)\} \in \mathcal{F}$.

So $M \leq n!$.

Hence

$$\sum_{F \in \mathcal{F}} |F|!(n - |F|)! \;\; \leq \;\; n!$$

$$1 \geq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \geq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = |\mathcal{F}| \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

Min-max statement for max-chains

A partition $\mathcal{C} = \{C_1, \ldots, C_l\}$ of P is a chain partition of P if all C_is are chains.

A partition $\mathcal{A} = \{A_1, \ldots A_k\}$ is an antichain partition of P if all A_is are antichains.

Proposition $\max\{|C| : C \text{ is a chain}\} =$
$\min\{|\mathcal{A}| : \mathcal{A} \text{ is an antichain partition of } P\}$

Proof. $\boxed{\leq}$ is immediate.
$\boxed{\geq}$ The set $A = \{x \in P : x \not\leq y \text{ for all } y \in P\}$ of maximum elements forms an antichain, that intersects every maximal chain of P.
So if P has maximum chain size M, then $P \setminus A$ has maximum chain size at most $M - 1$ (in fact equal).
By induction, find a partition of $P \setminus A$ into $M - 1$ antichains and extend it by A to get a partition of P into M antichains. □

Min-max statement for max-antichains

Dilworth's Theorem $\max\{|A| : A \text{ is an antichain}\} = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a chain partition of } P\}$

Proof. (Tverberg) $\boxed{\leq}$ is again immediate.
$\boxed{\geq}$ If there is a chain, that interesects every maximal antichain of P, then we proceed by induction as in the Proposition.
Otherwise let C be a maximal chain, that does not intersect the chain $A = \{a_1, \ldots, a_M\}$ of maximum size M. Let

$$\begin{aligned} A^- &= \{x \in P : x \leq a_i \text{ for some } i\} \\ A^+ &= \{x \in P : x \leq a_i \text{ for some } i\} \end{aligned}$$

- $A^- \cap A^+ = A$ because A is antichain
- $A^- \cup A^+ = P$ because A is maximal.

Apply induction on A^- and on A^+.

For this note that

$$A^- \neq P \;\Leftarrow\; \max C \in A^+ \setminus A \;\Leftarrow\; C \text{ is maximal}$$
$$A^+ \neq P \;\Leftarrow\; \min C \in A^- \setminus A \;\Leftarrow\; C \text{ is maximal}$$

Obtain

a chain partition $C_1^-, \ldots, C_M^-$ of A^- and
a chain partition $C_1^+, \ldots, C_M^+$ of A^+, such that
$C_i^- \cap A = \{a_i\} = C_i^+ \cap A$ for all i.

Then $C_1^- \cup C_1^+, \ldots, C_M^- \cup C_M^+$ is a partition of P into M chains. $\square$

Extremal set theory — the classics III

Proposition Let $\mathcal{F} \subseteq 2^{[n]}$ such that any two members of $\mathcal{F}$ have a nonempty intersection. Then

$$|\mathcal{F}| \leq 2^{n-1}.$$

Construction Proposition is best possible: Take all sets containing the element 1.

What if we restrict the sizes of the sets: all members must be of size k.
Taking all sets of size k that contains 1 gives $\binom{n-1}{k-1}$ sets. Is this again best possible?

Theorem (Erdős-Ko-Rado) Let $k, n \in \mathbb{N}$, $1 \leq k \leq n/2$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ such that any two members of $\mathcal{F}$ have a nonempty intersection. Then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Permutation method — reloaded

Proof. (Katona) C_n: set of cyclic permutations of $[n]$.

$$|C_n| = (n-1)!$$

Double-count
$M = |\{(\phi, F) : \phi \in C_n, F \in \mathcal{F} \text{ is a segment in } \phi\}|$

For $F \in \mathcal{F}$, let $C_F \subseteq C_n$ set of those cyclic permutations that contain F as a segment. $M = \sum_{F \in \mathcal{F}} |C_F|$.

$|C_F| = k!(n-k)! \Longrightarrow |\mathcal{F}|k!(n-k)! = M$.

Claim Every cyclic permutation can contain at most k different $F \in \mathcal{F}$ as a segment.

Claim $\Longrightarrow M \leq |C_n|k = (n-1)!k$.

$$\begin{aligned} |\mathcal{F}|k!(n-k)! &\leq (n-1)!k \\ |\mathcal{F}| &\leq \frac{(n-1)!k}{k!(n-k)!} \end{aligned}$$